Stationary Navier-Stokes Equations in two-dimensional Unbounded Domains

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde

(Dr.sc.nat.)

vorgelegt der

Mathematisch-naturwissenschaftlichen Fakultät

der Universität Zürich

von

Wei Xue

von

Werthenstein LU

Promotionskomitee

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Zürich 2016

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Acknowledgement

I give my sincere thanks to Prof. Dr. Michel Chipot for introducing the research topic to me and his guidance during my studies.

I would also like to thank the further members of my doctoral committee Prof. Dr. Stefan Sauter and Prof. Dr. Rémi Abgrall for their support.

I am also very grateful to the Institute of Mathematics of University of Zurich, where I also spent memorable time during my bachelor and master studies. I obtained constantly professional advice and encouragements by many members of the institute.

Finally, I give a special thanks to my grandfather XUE Bangxing who has never ceased to believe in me.

Abstract

In this thesis we study the nonhomogeneous boundary value problem for the stationary Navier-Stokes equations

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{u}+\big(\mathbf{u}\cdot\nabla\big)\mathbf{u}+\nabla p &=& \mathbf{f} & \text{in } \ \Omega, \\ \operatorname{div}\mathbf{u} &=& 0 & \text{in } \ \Omega, \\ \mathbf{u} &=& \mathbf{a} & \text{on } \ \partial\Omega \end{array}\right. \quad (0.1)$$

in two dimensional symmetric domains with finitely many outlets to infinity. The boundary of the domains may contain both finitely many inner or outer boundaries. We do not impose any restrictions on the size of the fluxes over the inner and outer boundaries. We show the existence of at least one weak solution to problem (0.1). Moreover, the Dirichlet integral of the solution can be either finite or infinite depending on the geometry of the domains.

We start with introducing this type of problems, the so called Leray problem. After giving a brief overview of the known results we recap some auxiliary theories which we use for the following chapters to prove our main results.

We show in the first part that if the given domain is non symmetric, problem (0.1) possesses at least one weak solution if the given boundary value satisfies the following flux condition

$$\int\limits_{\Gamma_j}\mathbf{a}\cdot\mathbf{n}\,dS=0, \qquad j=1,2,\ldots,N, \tag{0.2}$$

where Γ_j are the connected inner boundaries. Note that the flux over each connected outer boundary can be arbitrarily large.

We show then if the given domain and all the given data are symmetric with respect to the x_1-axis we can relax the above condition (0.2), i.e. only the necessary compatibility condition

$$0=\int\limits_{\Omega}\operatorname{div}\mathbf{u}\,dx=\int\limits_{\partial\Omega}\mathbf{a}\cdot\mathbf{n}\,dS=\sum_{j=1}^{N}\int\limits_{\Gamma_j}\mathbf{a}\cdot\mathbf{n}\,dS=\sum_{j=1}^{N}\mathbb{F}_j, \tag{0.3}$$

is assumed.

In both cases we construct a symmetric solenoidal extension of the boundary value satisfying the Leray-Hopf inequality. After having such an extension, the nonhomogeneous boundary value problem is reduced to the homogeneous one and the

existence of at least one weak solution follows.

In the last chapter we treat three problems on a two-dimensional "punctured periodic domain": $\Omega_r = (-L, L)^2 \setminus D_r$, where $D_r = B(0, r)$ is the disc of radius r centred at the origin. We impose periodic boundary conditions on the boundary of the box $\Omega = (-L, L)^2$, and Dirichlet boundary conditions on the circumference of the disc. In this setting we consider the Poisson equation, the Stokes equations, and the Navier-Stokes equations, all with a fixed forcing function $\mathbf{f}$, which must satisfy $\int_\Omega \mathbf{f} = 0$. We examine the behaviour of solutions as $r \to 0$. In the first two cases we show convergence of the solutions to those of the limiting problem, i.e. the problem posed on all of Ω with periodic boundary conditions. In the stationary Navier-Stokes case we analyze the failure of convergence to the solution of the limiting problem.

Zusammenfassung

In dieser Arbeit befassen wir uns mit dem Randwertproblem für die stationären Navier-Stokes Gleichungen

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{a} \quad \text{auf} \quad \partial\Omega \end{cases} \tag{0.4}$$

in zwei dimensionalen symmetrischen Gebieten. Der Rand der Gebiete ist zusammengesetzt aus beschränkt vielen inneren und äusseren Rändern. Wir setzen dabei keine Beschränkungen auf die Grösse der Flüsse über die inneren und äusseren Rändern voraus. Wir zeigen, dass mindestens eine schwache Lösung zum Problem (0.4) existiert. Desweiteren kann das Dirichlet Integral der Lösung in Abhängigkeit von der Geometrie des Gebietes entweder endlich oder unendlich sein.

Wir beginnen mit der Einführung von so genannten Leray Problemen. Nach einem kurzen Überblick über die bekannten Ergebnisse zeigen wir einige Hilfstheorien, die wir für unsere Beweise in den darauf folgenden Kapiteln brauchen.
Wir zeigen dann im ersten Teil, dass Problem (0.4) zumindest eine schwache Lösung besitzt, wenn der gegebene Randwert die folgende Bedingung erfüllt

$$\int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = 0, \qquad j = 1, 2, \ldots, N, \tag{0.5}$$

wobei Γ_j einen zusammenhängenden inneren Rand bezeichnet. Wir beachten dabei, dass der Fluss auf jedem zusammenhängenden äusseren Rand beliebig gross sein kann.
Wir zeigen danach, wenn das gegebene Gebiet und alle gegebenen Daten symmetrisch in Bezug auf die x_1-Achse sind können wir die obige Bedingung (0.5) abschwächen, d.h. nur die erforderliche Kompatibilitätsbedingung angenommen wird:

$$0 = \int\limits_{\Omega} \operatorname{div}\mathbf{u}dx = \int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dS = \sum_{j=1}^{N}\int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = \sum_{j=1}^{N} \mathbb{F}_j. \tag{0.6}$$

In beiden Fällen konstruieren wir eine symmetrische divergenzfreie Extension des Randwertes welche die Leray-Hopf Ungleichung erfüllt. Mit dieser Extension kann

man das nicht-homogene Randwertproblem auf das homogene Problem reduzieren, darauf folgt die Existenz mindestens einer schwachen Lösung.

Im letzten Kapitel behandeln wir drei Probleme auf einem zweidimensionalen "punktiert periodischen " Gebiet: $\Omega_r = (-L, L)^2 \setminus D_r$, wobei $D_r = B(0, r)$ die Scheibe mit Radius r und Mittelpunkt am Ursprung bezeichnet. Wir setzen periodische Randbedingungen auf dem Rand des Gebietes $\Omega = (-L, L)^2$, und Dirichlet-Randbedingung auf dem Rand der Scheibe voraus. Wir betrachten dann die Poisson-Gleichung, die Stokes-Gleichungen und die Navier-Stokes-Gleichungen, alle mit dem gegebenen Kraftfeld $\mathbf{f}$, welches die Bedingung $\int_\Omega \mathbf{f} = 0$ erfüllen muss. Wir untersuchen das Verhalten der Lösungen wenn r gegen null konvergiert. In den ersten beiden Fällen können wir die Konvergenz der Lösungen gegen die Lösung des Grenzproblemes, d.h. die Lösung auf dem ganzen Gebiet Ω mit periodischen Randbedingungen, zeigen. Für die stationären Navier-Stokes Gleichungen analysieren wir das Fehlverhalten der Konvergenz der Lösungen.

Chapter 1

Introduction

The Navier-Stokes equations describe the motion of an incompressible viscous fluid, which have been studied extensively for over one hundred years. We consider in this thesis the stationary nonhomogeneous boundary value problem of the Navier-Stokes equations:

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{a} \quad \text{on } \partial\Omega \end{cases} \tag{0.1}$$

in two dimensional symmetric[1] multiply connected domains Ω, having finitely many outlets to infinity, where the vector-valued function $\mathbf{u} = \mathbf{u}(x)$ is the unknown velocity field, the scalar function $p = p(x)$ is the pressure of the fluid, while the vector-valued functions $\mathbf{a} = \mathbf{a}(x)$ and $\mathbf{f} = \mathbf{f}(x)$ denote the given boundary value and the external force; $\nu > 0$ is the viscosity constant of the given fluid. The boundary $\partial\Omega$ consists of finitely many infinite connected outer boundaries and finitely many connected compact components, forming the inner boundaries. The fluxes of the boundary value $\mathbf{a}$ over each connected component of both inner and outer boundaries may be arbitrarily large.

1 Overview of the literature

The Navier-Stokes equations are of great interest in both physical and mathematical sense. The equations maybe used to model weather, water current or airflow. In the pure mathematical sense it has not yet been proven that there always exists solutions in three dimensions. The boundary and initial-boundary value problems for the Navier-Stokes equations have been also studied in many papers since the formulations by French mathematician J. Leray. The so called Leray problem remains

[1]For the definition of a symmetric domain see (1.1).

still open for the general case in both two dimensional and three dimensional space.

Bounded Domain

Let us start with the steady Navier-Stokes equations with nonhomogeneous boundary conditions

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{a} \quad \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

in a bounded domain Ω with Lipschitz boundary $\partial\Omega$ consisting of N disjoint components $\Gamma_j, j = 1, ..., N$. The incompressibility of the fluid ($\operatorname{div}\mathbf{u} = 0$) implies a necessary compatibility condition for the solvability of problem (1.1):

$$0 = \int_\Omega \operatorname{div}\mathbf{u}dx = \int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dS = \sum_{j=1}^N \int_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = \sum_{j=1}^N \mathbb{F}_j, \tag{1.2}$$

where $\mathbf{n}$ is a unit vector of the outward normal to $\partial\Omega$. For a long time the solvability of problem (1.1) was proved only under the condition

$$\mathbb{F}_j = \int_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = 0, \qquad j = 1, 2, \ldots, N, \tag{1.3}$$

(e.g., [37], [33], [34] or [64]).

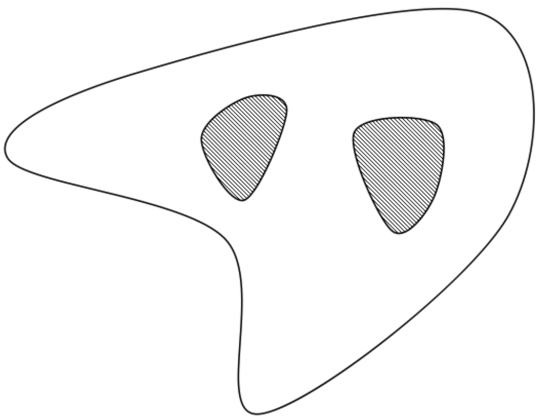


Figure 1.1: Multiply connected bounded domain Ω

This condition requires the flux $\mathbb{F}_j$ of the boundary value $\mathbf{a}$ to be zero on each connected component Γ_j of the boundary $\partial\Omega$. Clearly condition (1.3) implies (1.2), we thus call (1.3) sometimes stringent outflow condition and (1.2) the general out flow condition. The existence result can be also proven under the smallness assumptions on the fluxes $\mathbb{F}_j$ (e.g., [5], [10], [11], [16], [30]), or under certain symmetry assumptions on the domain Ω and the boundary value $\mathbf{a}$ (e.g., [1], [12], [13], [44], [49], [50], [54], [27]).

The Leray Problem

In 1933 J. Leray formulated the fundamental question whether problem (1.1) can be solved only under the necessary compatibility condition (1.2). This is so called Leray's problem which has been open for more than 80 years. The fundamental tool to solve the nonhomogeneous boundary value problem is to reduce it to the problem with homogeneous boundary conditions. Suppose that $\mathbf{A}$ is a solenoidal extension of the boundary value $\mathbf{a}$ into Ω such that

$$\mathbf{A}\big|_{\partial\Omega} = \mathbf{a}. \tag{1.4}$$

Moreover, $\mathbf{A}$ satisfies the so called Leray-Hopf inequality. We put then $\mathbf{u} = \mathbf{v}+\mathbf{A}$ into the first equation of (1.1) and look for the new unknown velocity field $\mathbf{v} \in \mathbb{W}_0^{1,2}(\Omega)$ satisfying the integral identity

$$\begin{aligned} &\nu \int_\Omega \nabla \mathbf{v} : \nabla \boldsymbol{\eta}\, dx - \int_\Omega ((\mathbf{A}+\mathbf{v})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{v}\, dx - \int_\Omega (\mathbf{v}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{A}\, dx \\ &= \int_\Omega (\mathbf{A}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{A}\, dx - \nu\int_\Omega \nabla\mathbf{A} : \nabla\boldsymbol{\eta}\, dx + \int_\Omega \mathbf{f}\cdot\boldsymbol{\eta}\, dx, \qquad \forall \boldsymbol{\eta} \in \mathbb{C}_0^\infty(\Omega). \end{aligned} \tag{1.5}$$

and the incompressibility condition:

$$\operatorname{div}\mathbf{v} = 0,$$

The existence of $\mathbf{v}$ satisfying (1.5) could be proved following the general scheme proposed by O. A. Ladyzhenskaya, we give later in this thesis a sketch of the proof in the Section **The Boundary Value Problem in Bounded Domains**. If the stringent outflow condition is satisfied one can always find such an extension $\mathbf{A}$ (see [33] or [16]). If only the general outflow condition is given, there is a counterexample (see [16]) showing that one cannot always find such a suitable extension for a general bounded domain.

Recently, Leray's problem was solved for a two dimensional multiply connected bounded domain ([26], [28], [29]). It is worth to mention that the method used there is not based on the usual technique which relies on constructing a suitable extension of the boundary value. Therefore, our method used in this thesis cannot apply to their result.

Unbounded Domain

However, Leray's problem remains still open for the unbounded domains, i.e., domains with outlets to infinity. For the general domains with outlets to infinity problem (1.1), (1.2) was solved (see [24], [25], [46], [47]) under the smallness assumption

of the fluxes over the bounded components of the boundary (notice that there are no restrictions on the fluxes over the infinite parts of the boundary).
Amick showed in [1] the existence result for two or three dimensional domains for the nonhomogeneous boundary case with two semi-infinite channels and under the assumption that the flux over each connected boundary equals to zero. Moreover, it is assumed that the solution converges to a certain Poisseuille flow at infinity. Rabier showed in [51] the existence of a symmetric solution for the two dimensional domain with two semi-infinite straight channels and under the assumption of zero flux over each connected component of the outer boundary. And there is no limitation on the magnitude of flux in the proof. For the non-symmetric case, Rabier showed the existence result for small flux.

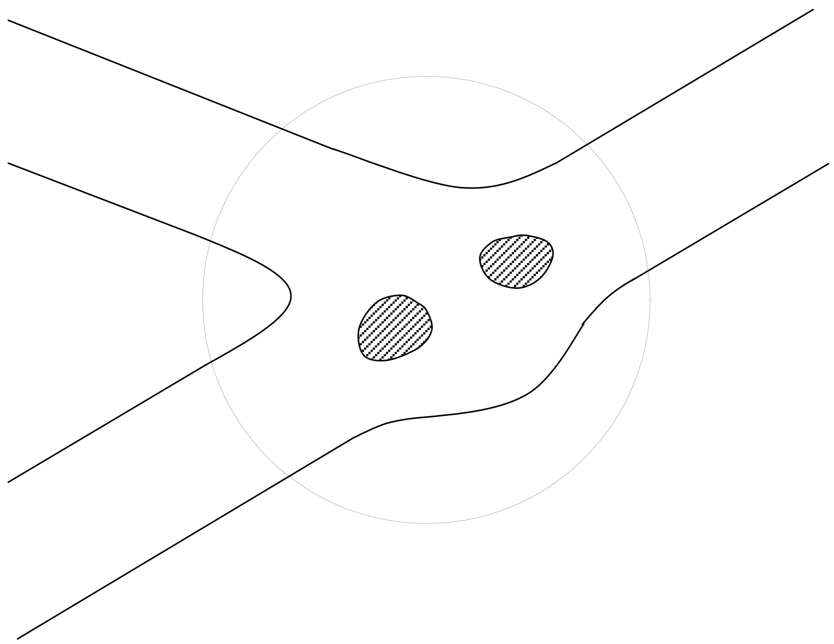


Figure 1.2: Domain Ω with channel-like outlets

Next, there is series of papers by H. Fujita and H. Morimoto (see [40]–[43]) where they solved problem (1.1) in symmetric two dimensional multiply connected domains Ω with channel-like outlets to infinity containing a finite number of inner boundaries under certain symmetry assumptions of the boundary value and external force. Moreover, in [40]–[43] the authors also assumed that the boundary value $\mathbf{a}$ is equal to zero on the outer boundary and that in each outlet the flow tends to a Poiseuille flow which needs to be sufficiently small, i.e., even the fluxes over each connected component of the inner boundary may be arbitrarily large, but the sum of them has to be sufficiently small.

In this thesis we consider problem (0.1) in general symmetric domains with outlets to infinity, i.e., the domain may have finitely many self-symmetric and pairwise symmetric outlets, assuming that the boundary value and external force are symmetric functions. We do not impose any restrictions on the fluxes over inner and outer boundaries.

2 Stucture of this thesis

This thesis is composed of two parts. In the first part, we study the nonhomogenous boundary value problem of Navier-Stokes equations in two dimensional symmetric unbounded domains having finitely many outlets. We firstly introduce the function spaces and recap some well-known results which we use later in the proof of our results. We then formulate our problem and introduce the technique of solving this problem. Following this technique we construct solenoidal extensions of boundary values in different types of domains. Finally, we show the existence results using the extensions of the boundary values.
In the second part we study the Stokes equation in a periodic domain. We are interested in fluid flow around a vanishing obstacle in a two-dimensional periodic domain. We start with studying the Poisson equation, and then the Stokes equations. In both cases we show convergence of the solutions to those of the limiting problem. In the last section we discuss the problems of applying the method on the steady Navier-Stokes equations.

Chapter 2

Leray's Problem in 2-D Unbounded Symmetric Domains

1 Main Notation and Auxiliary Results

Vector valued functions are denoted by bold letters while function spaces for vector valued functions are denoted by $\mathbb{L}^2, \mathbb{H}$ and so on.

Let Ω be a domain in $\mathbb{R}^n$. $\mathbb{C}^\infty(\Omega)$ denotes the set of all infinitely differentiable functions defined on Ω and $\mathbb{C}_0^\infty(\Omega)$ is the subset of all functions from $\mathbb{C}^\infty(\Omega)$ with compact support in Ω. For given nonnegative integers k and $q > 1$, $\mathbb{L}^q(\Omega)$ and $\mathbb{W}^{k,q}(\Omega)$ denote the usual Lebesgue and Sobolev spaces; $\mathbb{W}^{k-1/q,q}(\partial\Omega)$ is the trace space on $\partial\Omega$ of functions from $\mathbb{W}^{k,q}(\Omega)$; $\mathring{\mathbb{W}}^{k,q}(\Omega)$ is the closure of $\mathbb{C}_0^\infty(\Omega)$ with respect to the norm of $\mathbb{W}^{k,q}(\Omega)$; for an unbounded domain Ω we write $u \in \mathbb{W}^{k,q}_{loc}(\overline{\Omega})$ if $u \in \mathbb{W}^{k,q}(\Omega \cap B_R(0))$ for any $B_R(0) = \{x \in \mathbb{R}^2 : |x| \le R\}$.

Let $D(\Omega)$ be the Hilbert space of vector valued functions formed as the closure of $\mathbb{C}_0^\infty(\Omega)$ with respect to the Dirichlet norm $\|\mathbf{u}\|_{D(\Omega)} = \|\nabla\mathbf{u}\|_{L^2(\Omega)}$ induced by the scalar product

$$(\mathbf{u}, \mathbf{v}) = \int\limits_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, dx,$$

where $\nabla\mathbf{u} : \nabla\mathbf{v} = \sum\limits_{j=1}^n \nabla u_j \cdot \nabla v_j = \sum\limits_{j=1}^n \sum\limits_{k=1}^n \dfrac{\partial u_j}{\partial x_k}\dfrac{\partial v_j}{\partial x_k}$. Denote by $\mathbb{J}_0^\infty(\Omega)$ the set of all solenoidal ($\operatorname{div}\mathbf{u} = 0$) vector fields $\mathbf{u}$ from $\mathbb{C}_0^\infty(\Omega)$. By $\mathbb{H}(\Omega)$ we indicate the space formed as the closure of $\mathbb{J}_0^\infty(\Omega)$ with respect to the Dirichlet norm.

Assume that Ω is symmetric with respect to the x_1-axis, i.e.,

$$(x_1, x_2) \in \Omega \Leftrightarrow (x_1, -x_2) \in \Omega. \tag{1.1}$$

Definition 1.1. *The vector function $\mathbf{u} = (u_1, u_2)$ is called symmetric with respect to the x_1-axis if u_1 is an even function of x_2 and u_2 is an odd function of x_2, i.e.*

$$u_1(x_1, x_2) = u_1(x_1, -x_2), \qquad u_2(x_1, x_2) = -u_2(x_1, -x_2). \tag{1.2}$$

A function $\mathbf{u} = (u_1(x_1, x_2), u_2(x_1, x_2))$ *is called antisymmetric if*

$$u_1(x_1, -x_2) = -u_1(x_1, x_2), \qquad u_2(x_1, -x_2) = u_2(x_1, x_2). \tag{1.3}$$

For any set of functions $V(\Omega)$ defined in the symmetric domain Ω satisfying (1.1), we denote by $V_S(\Omega)$ the subspace of symmetric functions from $V(\Omega)$.

Lemma 1.1. *For vector valued functions* $\mathbf{u}$, $\mathbf{v}$, *the following equation holds true*

$$(\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} = \nabla(\mathbf{u} \cdot \mathbf{v}) - \omega(\mathbf{v})\mathbf{u}^{\perp} - \omega(\mathbf{u})\mathbf{v}^{\perp}.$$

where

$$\omega(\mathbf{v}) = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \mathbf{u}^{\perp} = (u_2, -u_1).$$

In particular,

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{2}\nabla|\mathbf{u}|^2 - \omega(\mathbf{u})\mathbf{u}^{\perp}.$$

Proof.

$$\begin{aligned}
&(\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \omega(\mathbf{v})\mathbf{u}^{\perp} + \omega(\mathbf{u})\mathbf{v}^{\perp} \\
&= (u_1\partial_1 + u_2\partial_2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + (v_1\partial_1 + v_2\partial_2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + (\partial_1 v_2 - \partial_2 v_1)\begin{pmatrix} u_2 \\ -u_1 \end{pmatrix} \\
&\quad + (\partial_1 u_2 - \partial_2 u_1)\begin{pmatrix} v_2 \\ -v_1 \end{pmatrix} \\
&= \begin{pmatrix} u_1\partial_1 v_1 + v_1\partial_1 u_1 + u_2\partial_1 v_2 + v_2\partial_1 u_2 \\ u_2\partial_2 v_2 + v_2\partial_2 u_2 + u_1\partial_2 v_1 + v_1\partial_2 u_1 \end{pmatrix} \\
&= \nabla(\mathbf{u} \cdot \mathbf{v}).
\end{aligned}$$

By substituting $\mathbf{v}$ by $\mathbf{u}$ we obtain:

$$\begin{aligned}
2(\mathbf{u} \cdot \nabla)\mathbf{u} &= \nabla(\mathbf{u} \cdot \mathbf{u}) - 2\omega(\mathbf{u})\mathbf{u}^{\perp} \\
\Rightarrow (\mathbf{u} \cdot \nabla)\mathbf{u} &= \frac{1}{2}\nabla|\mathbf{u}|^2 - \omega(\mathbf{u})\mathbf{u}^{\perp}.
\end{aligned}$$

□

Definition 1.2. *Suppose that the domain* Ω *is symmetric with respect to the* x_1*-axis. For a vector field* $\varphi = \varphi(x_1, x_2)$ *defined in* Ω, *define the symmetric part* φ^s *of* φ *as follows,*

$$\varphi^s(x_1, x_2) = \frac{1}{2}(\varphi_1(x_1, x_2) + \varphi_1(x_1, -x_2), \varphi_2(x_1, x_2) - \varphi_2(x_1, -x_2)).$$

and the antisymmetric part φ^a *of* φ *is*

$$\varphi^a(x_1, x_2) = \frac{1}{2}(\varphi_1(x_1, x_2) - \varphi_1(x_1, -x_2), \varphi_2(x_1, x_2) + \varphi_2(x_1, -x_2)).$$

$\varphi = \varphi^s + \varphi^a$ *holds true.*

Definition 1.3. *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *be smooth functions defined in* $\overline{\Omega}$. *We define a trilinear form as follows*

$$((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w})_{L^2(\Omega)} := \int\limits_{\Omega} \sum_{i,j=1}^{2} u_i \frac{\partial v_j}{\partial x_i} \omega_j dx. \tag{1.4}$$

Lemma 1.2.

$$((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w}) = \int_{\partial\Omega}(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{w})dS - \int_{\Omega}(\mathbf{w}\cdot\mathbf{v})\operatorname{div}\mathbf{u}dx - ((\mathbf{u}\cdot\nabla)\mathbf{w}, \mathbf{v}) \tag{1.5}$$

Where $\mathbf{n}$ *is the unit outward normal vector to the boundary* $\partial\Omega$. *Furthermore,*

$$\begin{aligned}&((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w}) = -((\mathbf{u}\cdot\nabla)\mathbf{w}, \mathbf{v}), \quad \textit{for} \;\; \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{J}_0^\infty,\\ &\textit{and}\; ((\mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{v}) = 0 \; \textit{for}\; \mathbf{u}, \mathbf{v} \in \mathbb{J}_0^\infty.\end{aligned}$$

Proof.

$$\begin{aligned}
&((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w}) + ((\mathbf{u}\cdot\nabla)\mathbf{w}, \mathbf{v})\\
&= \textstyle\sum_{i,j=1}^{2} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j + u_i \frac{\partial w_j}{\partial x_i} v_j\\
&= \textstyle\sum_{i,j=1}^{2} \int_{\Omega} u_i \frac{\partial}{\partial x_i}(v_j w_j)\\
&= \textstyle\sum_{i,j=1}^{2} \int_{\Omega} \frac{\partial}{\partial x_i}(u_i v_j w_j) - \frac{\partial u_i}{\partial xi}(v_j w_j)\\
&= \textstyle\sum_{i,j=1}^{2} \int_{\partial\Omega} u_i v_j w_j n_i - \int_{\Omega} \frac{\partial u_i}{\partial x_i}(v_j w_j)\\
&= \textstyle\sum_{i,j=1}^{2} \int_{\partial\Omega}(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{w}) - \int_{\Omega}\operatorname{div}\mathbf{u}(\mathbf{v}\cdot\mathbf{w}).
\end{aligned}$$

□

Below we show some well known results which are formulated in the following lemmas.

Lemma 1.3. *(Leray-Schauder theorem). Let* V *be a Hilbert space and* $\mathcal{A}: V \to V$ *be a nonlinear compact operator. If the norms of all possible solutions of the operator equation*

$$u^{(\lambda)} = \lambda\mathcal{A}u^{(\lambda)}, \quad \lambda \in [0,1],$$

are bounded with the same constant c *independent of* λ, *i.e.,*

$$\|u^{(\lambda)}\|_V \leq c \quad \forall\lambda \in [0,1],$$

then the operator equation

$$u = \mathcal{A}u$$

has at least one solution $u \in V$ *(see, for example, [33]).*

Definition 1.4. *We call a kernel $K(x,y)$ of the form $K(x,y)=\frac{k(x,y)}{|y|^n}$ singular with $k(x,y)$ a regular function, $x\in\Omega$, $y\in\mathbb{R}^n\setminus\{0\}$ if and only if*
(i) For any x, y and every $\alpha>0$

$$k(x,y)=k(x,\alpha y);$$

(ii) For every $x\in\Omega$, $k(x,y)$ is integrable on the sphere $|y|=1$ and

$$\int\limits_{|y|=1} k(x,y)d\sigma_y=0;$$

(iii) For some $q>1$, there exists $C>0$ such that

$$\int\limits_{|y|=1} |k(x,y)|^q d\sigma_y\le C, \quad \text{uniformly in } x.$$

Lemma 1.4. *(See [2]) Let $\Omega\subset\mathbb{R}^n$, $n=2,3$ be a bounded Lipschitz domain. Let $1<p<\infty$, $f\in L^p(\Omega)$ and $\int_\Omega f=0$. Then the following divergence problem*

$$\begin{cases}\operatorname{div}\mathbf{v}=f & \text{in } \Omega\\ \mathbf{v}\in W_0^{1,p}(\Omega)\end{cases}\tag{1.6}$$

admits a solution f with

$$\|\mathbf{v}\|_{\mathbb{W}_0^{1,p}(\Omega)}\le C\|f\|_{L^p(\Omega)},$$

where the constant C depends on p, R and the diameter of Ω.

We notice that $\int_\Omega f=0$ is a necessary condition as a consequence of the system (1.6). This lemma can be proved following the idea of Bogovskii (see [4]), which gives an explicit solution of the above problem. In fact the regularity of the domain Ω is important. Ω need to be decomposed to finitely many subdomains Ω_k such that

$$\Omega=\bigcup_{k=1}^{N}\Omega_k,$$

where every Ω_k is star-liked with respect to an open ball B_k with $\overline{B}_k\subset\Omega_k$. This condition can be satisfied if the domain is for example locally Lipschitz. Then we can assume that our domain is star-like. Assume then first $f\in C_0^\infty(\Omega)$. By the change of variables

$$x'=\frac{x-x_0}{R},$$

the point x_0 is shifted to the origin of coordinates and $B_R(x_0)$ is transformed into $B_1(0)\equiv B$. Moreover, Ω goes into a domain Ω' that is star-shaped with respect to every point of B with

$$\delta(\Omega')=\delta(\Omega)/R.$$

Where $\delta(\Omega)$ is the diameter of Ω, that is,

$$\delta(\Omega) = \sup_{x,y\in\Omega} |x-y|.$$

While $\mathbf{v}$ goes into $\mathbf{v}'$, f into f' equation (1.6) becomes

$$\operatorname{div}\mathbf{v}' = Rf' \equiv F' \text{ in } \Omega', \tag{1.7}$$

satisfying $\int_{\Omega'} F' = 0$ and $F' \in C_0^\infty(\Omega')$. Furthermore, if $\mathbf{v}'$, F' verify (1.7), the transformed functions $\mathbf{v}$ and f through the inverse of the function changing of variables verify (1.6). Let ω be a function in $C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned} &(i) \quad supp(\omega) \subset B,\\ &(ii) \quad \int_B \omega = 1. \end{aligned}$$

We can show that the vector field

$$\begin{aligned} \mathbf{v}(x) &= \int_\Omega F(y)\left[\frac{x-y}{|x-y|^n}\int_{|x-y|}^{\infty} \omega\Big(y+\xi\frac{x-y}{|x-y|}\Big)\xi^{n-1}d\xi\right]dy\\ &\equiv \int_Q F(y)N(x,y)dy \end{aligned} \tag{1.8}$$

solves (1.7). Finally, we can approximate $f \in L^q(\Omega)$ by a sequence $\{f_n\} \subset C_0^\infty(\Omega)$.

Remark 1.1. *Similarly one can show the existence of at least one solution to problem* (1.6) *for the nonhomogeneous case:*

$$\begin{cases} \nabla\cdot\mathbf{v} = f,\\ \mathbf{v} \in \mathbb{W}^{1,q}(\Omega),\\ \mathbf{v} = \mathbf{a} \ \textit{on} \ \partial\Omega, \end{cases}$$

given $f \in L^q$, $\mathbf{a} \in \mathbb{W}^{1-1/q,q}(\Omega)$, $1 < q < \infty$ *and satisfying*

$$\int_\Omega f = \int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}.$$

Lemma 1.5. *(see [33]) Let* $\Pi \subset \mathbb{R}^2$ *be a bounded domain with Lipschitz boundary* $\partial\Pi$. *Then for any* $\mathbf{w} \in \mathbb{W}^{1,2}(\Pi)$ *with* $\mathbf{w}\big|_{\mathcal{L}} = 0$, *a subset* $\mathcal{L} \subseteq \partial\Pi$, *the following inequality*

$$\int_\Pi \frac{|\mathbf{w}|^2\,dx}{\operatorname{dist}^2(x,\mathcal{L})} \le c\int_\Pi |\nabla\mathbf{w}|^2\,dx \tag{1.9}$$

holds.

Lemma 1.6. *Let Ω be a symmetric domain with respect to the x_1-axis. Let $\mathbf{u} \in \mathbb{L}^2(\Omega)$ be symmetric, and $\mathbf{v} \in \mathbb{L}^2(\Omega)$ antisymmetric. Then,*

$$(\mathbf{u}, \mathbf{v})_{L^2(\Omega)} = \int\limits_{\Omega} \mathbf{u}(x) \cdot \mathbf{v}(x) dx = 0. \tag{1.10}$$

Furthermore, if $\mathbf{u}, \mathbf{v} \in \mathbb{H}^1(\Omega)$, then,

$$(\nabla \mathbf{u}, \nabla \mathbf{v})_{L^2(\Omega)} = 0. \tag{1.11}$$

Proof. Define

$$\Omega_+ = \{x = (x_1, x_2) \in \Omega | x_2 > 0\}, \quad \Omega_- = \{x = (x_1, x_2) \in \Omega | x_2 < 0\}.$$

$$\mathbf{u}(x) \cdot \mathbf{v}(x) = u_1(x_1, x_2) v_1(x_1, x_2) + u_2(x_1, x_2) v_2(x_1, x_2).$$

Since $u_1(x_1, -x_2) = u_1(x_1, x_2)$, $u_2(x_1, -x_2) = -u_2(x_1, x_2)$, $v_1(x_1, -x_2) = -v_1(x_1, x_2)$ and $v_2(x_1, -x_2) = v_2(x_1, x_2)$ we have

$$u_1(x_1, -x_2) v_1(x_1, -x_2) = -u_1(x_1, x_2) v_1(x_1, x_2).$$

Therefore:

$$\begin{aligned}\int\limits_{\Omega} u_1(x) v_1(x) dx &= \int\limits_{\Omega_+} u_1(x) v_1(x) + \int\limits_{\Omega_-} u_1(x) v_1(x) dx \\ &= \int\limits_{\Omega_+} u_1(x) v_1(x) - \int\limits_{\Omega_+} u_1(x) v_1(x) dx = 0.\end{aligned}$$

Because Ω is symmetric with respect to the x_1-axis. Similarly, we get

$$\int\limits_{\Omega} u_2(x) v_2(x) = 0.$$

Thus, the first part of the lemma is proven. The second part can be proven in the same way. $\square$

Lemma 1.7. *Let Ω be a symmetric domain with respect to the x_1-axis. Let $\mathbf{u} \in \mathbb{H}^1(\Omega)$ be symmetric with respect to the x_1-axis. Then the trace to the x_1-axis exists and the second component vanishes there, that is,*

$$\textit{for } (x_1, 0) \in \Omega \quad \mathbf{u}(x_1, 0) = (u_1(x_1, 0), u_2(x_1, 0)) = (u_1(x_1, 0), 0).$$

Proof. Since $\mathbf{u}$ is symmetric, we have $u_2(x_1, 0) = -u_2(x_1, 0)$. Therefore, $u_2 = 0$. $\square$

Lemma 1.8. *Let Ω be a symmetric domain with respect to the x_1-axis. Define*

$$\Omega_+ = \{x = (x_1, x_2) \in \Omega | x_2 > 0\},\ \Omega_- = \{x = (x_1, x_2) \in \Omega | x_2 < 0\} \text{ and}$$
$$\Gamma_{i,+} = \{x = (x_1, x_2) \in \Gamma_i | x_2 > 0\},\ \Gamma_{i,-} = \{x = (x_1, x_2) \in \Gamma_i | x_2 < 0\},\ i = 0, \dots, N.$$

Let $\mathbf{u} \in \mathbb{H}^1(\Omega)$ be symmetric with respect to the x_1-axis. Then, $\mathbf{u} \cdot \mathbf{n}$ is even in x_2 and

$$\int\limits_{\Gamma_{i,+}} \mathbf{u} \cdot \mathbf{n} dS = \int\limits_{\Gamma_{i,-}} \mathbf{u} \cdot \mathbf{n} dS = \frac{1}{2} \int\limits_{\Gamma_i} \mathbf{u} \cdot \mathbf{n} dS \text{ for } i = 0, \dots, N$$
$$\int\limits_{\partial\Omega_+} \mathbf{u} \cdot \mathbf{n} dS = \sum_{i=0}^{N} \int\limits_{\Gamma_{i,+}} \mathbf{u} \cdot \mathbf{n} dS.$$

Proof. Since the domain Ω is symmetric with respect to the x_1-axis, the outward normal vector $\mathbf{n}$ is also symmetric. Thus,

$$\begin{aligned} \mathbf{u}(x_1, -x_2) \cdot \mathbf{n}(x_1, -x_2) &= (u_1(x_1, -x_2), u_2(x_1, -x_2)) \cdot (n_1(x_1, -x_2), n_2(x_1, -x_2)) \\ &= (u_1(x_1, x_2), -u_2(x_1, x_2)) \cdot (n_1(x_1, x_2), -n_2(x_1, x_2)) \\ &= (u_1(x_1, x_2), u_2(x_1, x_2)) \cdot (n_1(x_1, x_2), (n_2(x_1, x_2)) \end{aligned}$$

Because the domain Ω is symmetric and $\mathbf{u} \cdot \mathbf{n}$ is even, for $i = 0, \dots, N$ we have

$$\begin{aligned} &\int\limits_{\Gamma_{i,+}} \mathbf{u} \cdot \mathbf{n} dS = \int\limits_{\Gamma_{i,+}} (u_1(x_1, x_2), u_2(x_1, x_2)) \cdot (n_1(x_1, x_2), n_2(x_1, x_2)) dS \\ &= \int\limits_{\Gamma_{i,-}} (u_1(x_1, -x_2), u_2(x_1, -x_2)) \cdot (n_1(x_1, -x_2), n_2(x_1, -x_2)) dS \\ &= \int\limits_{\Gamma_{i,-}} (u_1(x_1, x_2), u_2(x_1, x_2)) \cdot (n_1(x_1, x_2), n_2(x_1, x_2)) dS = \int\limits_{\Gamma_{i,-}} \mathbf{u} \cdot \mathbf{n} dS \\ &= \frac{1}{2} \int\limits_{\Gamma_i} \mathbf{u} \cdot \mathbf{n} dS \end{aligned}$$

□

Lemma 1.9. *(see [33]) Let $\Pi \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Pi$, a subset $\mathcal{L} \subseteq \partial\Pi$ and the function $\mathbf{h} \in \mathbb{W}^{1/2,2}(\partial\Pi)$ satisfies the conditions $\int\limits_{\mathcal{L}} \mathbf{h} \cdot \mathbf{n} \, dS = 0$, $\operatorname{supp} \mathbf{h} \subseteq \mathcal{L}$. Then $\mathbf{h}$ can be extended inside Π in the form*

$$\mathbf{b}_0(x, \varepsilon) = \left(\frac{\partial(\chi(x, \varepsilon) \cdot E(x))}{\partial x_2}, -\frac{\partial(\chi(x, \varepsilon) \cdot E(x))}{\partial x_1} \right), \tag{1.12}$$

where $E \in W^{2,2}(\Pi)$, $\left(\frac{\partial E(x)}{\partial x_2}, -\frac{\partial E(x)}{\partial x_1} \right)\Big|_{\partial\Pi} = \mathbf{h}$ and χ is a Hopf's type cut-off function, i.e., χ is smooth, $\chi(x, \varepsilon) = 1$ on $\mathcal{L}$, $\operatorname{supp} \chi$ is contained in a small neighborhood *of* $\mathcal{L}$ *and*

$$|\nabla\chi(x,\varepsilon)| \leq \frac{\varepsilon\, c}{\mathrm{dist}(x,\mathcal{L})}.$$

The constant c *is independent of* ε.

Let $\mathcal{M}$ be a closed set in $\mathbb{R}^2$. $\Delta_{\mathcal{M}}(x)$ denotes the regularized distance from the point x to the set $\mathcal{M}$. Notice that $\Delta_{\mathcal{M}}(x)$ is an infinitely differentiable function in $\mathbb{R}^2 \setminus \mathcal{M}$ and the following inequalities

$$\begin{aligned} &a_1 d_{\mathcal{M}}(x) \leq \Delta_{\mathcal{M}}(x) \leq a_2 d_{\mathcal{M}}(x), \\ &|D^\alpha \Delta_{\mathcal{M}}(x)| \leq a_3 d_{\mathcal{M}}^{1-|\alpha|}(x) \end{aligned} \tag{1.13}$$

hold, where $d_{\mathcal{M}} = dist(x,\mathcal{M})$ is the distance from x to $\mathcal{M}$, the positive constants a_1, a_2 and a_3 are independent of $\mathcal{M}$ (see [59]).

2 Problem Formulation and Solvability

2.1 Formulation of the Problem

Let $\Omega \subset \mathbb{R}^2$ be an unbounded symmetric domain

$$\Omega = \Omega_0 \cup D_1 \cup ... \cup D_N, \quad D_j \cap D_k = \emptyset, \quad j \neq k,$$

where $\Omega_0 = \Omega \cap B_{R_0}(0) \subset B_{R_0}(0)$ is the bounded part of the domain Ω and the unbounded components D_j, $j = 1, ..., N$, are called "outlets to infinity". These outlets D_j in some cartesian coordinate systems $z^{(j)}$ have the form

$$D_j = \{z^{(j)} \in \mathbb{R}^2 : |z_2^{(j)}| < g_j(z_1^{(j)}),\ z_1^{(j)} > R_0\},$$

where $z^{(j)}$ means the local coordinate system in the outlet D_j and $g_j(t) > 0$ are functions satisfying the Lipschitz condition

$$|g_j(t_1) - g_j(t_2)| \leq L_j|t_1 - t_2|, \quad t_1, t_2 \geq R_0.$$

Depending on the function g_j each outlet D_j may expand at infinity but not too much in order not to intersect each other. Notice that if the g_j is a constant function, then we have channel-like outlet. Therefore, channel-like outlets are included as well.

Since we consider symmetric domains we may have pairs of outlets which are symmetric to each other (briefly symmetric outlets) and self-symmetric outlets.

Definition 2.5. *We call a domain* $\Omega \subset \mathbb{R}^2$ *an admissible domain if* Ω *satisfies the following assumptions*

(i) the bounded domain Ω_0 *has the form*

$$\Omega_0 = G_0 \setminus \cup_{i=1}^{I} G_i,$$

where G_0 and G_i, $\;i = 1, ..., I$, are bounded simply connected domains such that $\overline{G}_i \subset G_0$. Denote $\partial G_i = \Gamma_i$. Each Γ_i, $i = 1, ..., I$, intersects the x_1- axis;
(ii) the boundary $\partial\Omega$ is Lipschitz and it is composed of the inner boundaries $\cup_{i=1}^{I}\Gamma_i = \Gamma$ and the outer boundary $\partial\Omega \setminus \Gamma = \Gamma_0$. The outer boundary Γ_0 consists of not connected unbounded components Γ_0^m, $\;m = 1, ..., N$, i.e., $\cup_{m=1}^{N}\Gamma_0^m = \Gamma_0$.

Remark 2.2. Notice that as far as the outlets do not intersect each other the domain Ω in general may have finitely many outlets to infinity.

Below we use the following notations for $j = 1, \ldots, N$:

$$R_{j,0} = R_0, \quad R_{j,l+1} = R_{j,l} + \frac{g_j(R_{j,l})}{2L_j}, \; l \geq 1,$$
$$D_j^{(l)} = \{z^{(j)} \in D_j : z_1^{(j)} < R_{j,l}\}, \quad \Omega_l = \Omega_0 \cup D_1^{(l)} \cup ... \cup D_N^{(l)},$$
$$\omega_l = \Omega_{l+1} \setminus \overline{\Omega}_l \; .$$

Remark 2.3. There holds the relation for $j = 1, \ldots, N$

$$\frac{1}{2}g_j(R_{j,k}) \leq g_j(t) \leq \frac{3}{2}g_j(R_{j,k}), \quad t \in [R_{j,k}, \; R_{j,k+1}]. \tag{2.1}$$

Indeed, for $t \in [R_{j,k}, \; R_{j,k+1}]$ one has

$$-\frac{g_j(R_{j,k})}{2} = -L_j\left(R_{j,k+1} - R_{j,k}\right) \leq -L_j\left(t - R_{j,k}\right) \leq g_j(t) - g_j(R_{j,k})$$

$$\leq L_j\left(t - R_{j,k}\right) \leq L_j\left(R_{j,k+1} - R_{j,k}\right) = L_j\,\frac{g_j(R_{j,k})}{2L_j}.$$

This implies (2.1).

We consider the following problem

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p & = & \mathbf{f} & \text{in } \;\Omega, \\ \operatorname{div}\mathbf{u} & = & 0 & \text{in } \;\Omega, \\ \mathbf{u} & = & \mathbf{a} & \text{on } \;\partial\Omega \\ \int\limits_{\sigma_j(R)} \mathbf{u}\cdot\mathbf{n}\,dS & = & \mathbb{F}_j, & j = 1, ..., N, \; R \geq R_0, \end{array}\right. \tag{2.2}$$

where $\mathbb{F}_j$, $j = 1, ..., N$, are the prescribed fluxes over the cross sections $\sigma_j(R)$ of the outlets D_j, $\mathbf{n}$ is the unit normal to σ_j.
We suppose that the boundary value $\mathbf{a} \in \mathbb{W}^{1/2,2}(\partial\Omega)$ has a compact support and we denote $\Lambda^m := \operatorname{supp}\mathbf{a} \subset (\partial\Omega_0 \cap \Gamma_0^m)$, $m = 1, ..., N$. Let

$$\int\limits_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\,dS := \mathbb{F}_i^{(inn)}, \; i = 1, ..., I \qquad \int\limits_{\Lambda^m} \mathbf{a}\cdot\mathbf{n}\,dS := \mathbb{F}_m^{(out)}, \quad m = 1, ..., N$$

be the fluxes of the boundary value $\mathbf{a}$ over the inner and outer boundaries, respectively. Denote $\sum\limits_{i=1}^{I}\mathbb{F}_i^{(inn)} := \mathbb{F}^{(inn)}$ and $\sum\limits_{m=1}^{N}\mathbb{F}_m^{(out)} := \mathbb{F}^{(out)}$.
Notice that for Λ^j and Λ^k, $j \neq k$, which are symmetric to each other, we have

$$\mathbb{F}_j^{(out)} = \int\limits_{\Lambda^j} \mathbf{a}\cdot\mathbf{n}\,dS = \int\limits_{\Lambda^k} \mathbf{a}\cdot\mathbf{n}\,dS = \mathbb{F}_k^{(out)}.$$

2.2 Solvability of Problem (2.2)

Definition 2.6. *Under a symmetric weak solution of problem* (2.2) *we understand a solenoidal vector field* $\mathbf{u} \in \mathbb{W}^{1,2}_{loc,S}(\overline{\Omega})$ *satisfying the boundary condition* $\mathbf{u}\big|_{\partial\Omega} = \mathbf{a}$, *the flux conditions*

$$\int\limits_{\sigma_j(R)} \mathbf{u} \cdot \mathbf{n} \, dS = \mathbb{F}_j, \quad j = 1, ..., N, \quad R > R_0$$

and the integral identity

$$\nu \int\limits_{\Omega} \nabla \mathbf{u} : \nabla \boldsymbol{\eta} \, dx - \int\limits_{\Omega} (\mathbf{u} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{u} \, dx = \int\limits_{\Omega} \mathbf{f} \cdot \boldsymbol{\eta} \, dx, \qquad \forall \boldsymbol{\eta} \in \mathbb{J}^{\infty}_{0,S}(\Omega), \tag{2.3}$$

where $\mathbb{J}^{\infty}_{0,S}(\Omega)$ *denotes all solenoidal functions in* $\mathbb{C}^{\infty}_0(\Omega)$.

General scheme of showing existence of a weak solution

Let $\mathbf{A}$ be a symmetric solenoidal extension of the boundary value $\mathbf{a}$ into Ω such that

$$\mathbf{A}\big|_{\partial\Omega} = \mathbf{a}, \qquad \int\limits_{\sigma_j(R)} \mathbf{A} \cdot \mathbf{n} \, dS = \mathbb{F}_j, \quad j = 1, ..., N. \tag{2.4}$$

Let $\mathbf{f}$ be a symmetric vector field, put $\mathbf{u} = \mathbf{v} + \mathbf{A}$ into identity (2.3) and look for the new unknown velocity field $\mathbf{v} \in \mathbb{W}^{1,2}_{loc,S}(\overline{\Omega})$ satisfying the integral identity

$$\begin{aligned} &\nu \int\limits_{\Omega} \nabla \mathbf{v} : \nabla \boldsymbol{\eta} \, dx - \int\limits_{\Omega} ((\mathbf{A} + \mathbf{v}) \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{v} \, dx - \int\limits_{\Omega} (\mathbf{v} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A} \, dx \\ &= \int\limits_{\Omega} (\mathbf{A} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A} \, dx - \nu \int\limits_{\Omega} \nabla \mathbf{A} : \nabla \boldsymbol{\eta} \, dx + \int\limits_{\Omega} \mathbf{f} \cdot \boldsymbol{\eta} \, dx, \qquad \forall \boldsymbol{\eta} \in \mathbb{J}^{\infty}_{0,S}(\Omega). \end{aligned} \tag{2.5}$$

and zero boundary and flux conditions:

$$\operatorname{div} \mathbf{v} = 0, \quad \mathbf{v}\big|_{\partial\Omega} = 0, \qquad \int\limits_{\sigma_j(R)} \mathbf{v} \cdot \mathbf{n} \, dS = 0, \quad j = 1, ..., N, \quad R > R_0 > 0.$$

Remark 2.4. Notice that a symmetric weak solution to the integral identity (2.5) remains valid for any non-symmetric test functions $\boldsymbol{\eta} \in \mathbb{J}^{\infty}_0(\Omega)$. It is well known that each function $\boldsymbol{\eta} \in \mathbb{J}^{\infty}_0(\Omega)$ can be decomposed to a sum $\boldsymbol{\eta} = \boldsymbol{\eta}^s + \boldsymbol{\eta}^a$, where $\boldsymbol{\eta}^s$ is symmetric and $\boldsymbol{\eta}^a$ is antisymmetric, and because of Lemma 1.6 that all integrals in (2.5) vanish for $\boldsymbol{\eta} = \boldsymbol{\eta}^a$.

The existence of $\mathbf{v}$ satisfying (2.5) could be proved following the general scheme proposed by V. A. Solonnikov [58]. We give the general idea of the existence proof. Let us assume (as in [58]) that there is a sequence of bounded domains Ω_l such that $\Omega_l \subset \Omega_{l+1}$ and Ω_l exhausts Ω as $l \to +\infty$. We construct a solution to (2.5) as a limit of sequence $\mathbf{v}^{(l)} \in \mathbb{H}_S(\Omega_l)$, where $\mathbf{v}^{(l)}$ are the solutions to

$$\begin{aligned} &\nu \int\limits_{\Omega_l} \nabla \mathbf{v}^{(l)} : \nabla \boldsymbol{\eta} \, dx - \int\limits_{\Omega_l} ((\mathbf{A} + \mathbf{v}^{(l)}) \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{v}^{(l)} \, dx - \int\limits_{\Omega_l} (\mathbf{v}^{(l)} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A} \, dx \\ &= \int\limits_{\Omega_l} (\mathbf{A} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A} \, dx - \nu \int\limits_{\Omega_l} \nabla \mathbf{A} : \nabla \boldsymbol{\eta} \, dx + \int\limits_{\Omega} \mathbf{f} \cdot \boldsymbol{\eta} \, dx, \qquad \forall \boldsymbol{\eta} \in \mathbb{J}^{\infty}_{0,S}(\Omega_l). \end{aligned} \tag{2.6}$$

This integral identity is equivalent to the operator equation:

$$\mathbf{v}^{(l)} = \mathcal{A}\,\mathbf{v}^{(l)} \tag{2.7}$$

with the compact operator $\mathcal{A}$ in the space $H_S(\Omega_l)$. The solvability of the operator equation (2.7) can be obtained by applying the Leray-Schauder theorem, i.e., we need to show that all possible solutions of the operator equation

$$\mathbf{v}^{(l,\lambda)} = \lambda\mathcal{A}\,\mathbf{v}^{(l,\lambda)}, \quad \lambda \in [0,1], \tag{2.8}$$

are uniformly (with respect to λ) bounded. To do this we need to construct an extension $\mathbf{A}$ satisfying the following so called Leray-Hopf inequality:

$$\Big|\int\limits_{\Omega_l}(\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{A}\,dx\Big| \le c\,\varepsilon\int\limits_{\Omega_l}|\nabla\mathbf{w}|^2\,dx \tag{2.9}$$

for every symmetric solenoidal function $\mathbf{w} \in \mathbb{W}^{1,2}_{loc}(\overline{\Omega})$ with $\mathbf{w}|_{\partial\Omega} = 0$.
If (2.9) is true, then we obtain the following estimate:

$$\int\limits_{\Omega_l}|\nabla\mathbf{u}|^2\,dx \le c(\mathbf{a},\|\mathbf{f}\|_*)\left(1+\sum_{j=1}^{N}\int\limits_{R_0}^{R_l}\frac{dx_1}{g_j^3(x_1)}\right), \tag{2.10}$$

where the constant $c(\mathbf{a},\|\mathbf{f}\|_*)$ is defined below in the Theorem 2.1.
If $\int\limits_{R_0}^{R_l}\frac{dx_1}{g_j^3(x_1)} < +\infty$ for every $j = 1,\dots,N$, then the right hand side of (2.10) is bounded by a constant uniformly independent of l and we get for a limit vector function the integral identity (2.5).
If there exists at least one number j such that $\int\limits_{R_0}^{R_l}\frac{dx_1}{g_j^3(x_1)} = +\infty$, then we cannot pass to a limit because the right hand side of (2.10) is growing. Therefore, we need to control the Dirichlet integral of $\mathbf{v}^{(l)}$ over subdomains $\Omega_k \subset \Omega_l,\ \ k \le l$. To do this we need to apply the special techniques (so called estimates of Saint Venant type) developed by V.A. Solonnikov and O. A. Ladyzhenskaya (see [36], [58]). In order to get a suitable estimate we need to construct an extension $\mathbf{A}$ satisfying the Leray-Hopf inequality (2.9) and

$$\Big|\int\limits_{\Omega_{k+1}\setminus\Omega_k}(\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{A}\,dx\Big| \le c\,\varepsilon\int\limits_{\Omega_{k+1}\setminus\Omega_k}|\nabla\mathbf{w}|^2\,dx \tag{2.11}$$

for every symmetric solenoidal function $\mathbf{w} \in \mathbb{W}^{1,2}_{loc}(\overline{\Omega})$ with $\mathbf{w}|_{\partial\Omega} = 0$. Then we obtain the following estimate:

$$\int\limits_{\Omega_k}|\nabla\mathbf{u}|^2\,dx \le c(\mathbf{a},\|\mathbf{f}\|_*)\left(1+\sum_{j=1}^{N}\int\limits_{R_0}^{R_{j,k}}\frac{dx_1}{g_j^3(x_1)}\right), \quad k \le l.$$

This estimate ensures the existence of a subsequence $\{\mathbf{v}^{(l_m)}\}$ which converges weakly in (Ω_k) and strongly in $L_4(\Omega_k)$, $\ \forall k > 0$, and we can pass to a limit as $l_m \to \infty$. As a result we get for a limit vector function the integral identity (2.5).

Remark 2.5. *The symmetric assumptions are only necessary for the construction of a solenoidal extension of the inner boundary value having arbitrary big flux. The technique for showing the existence of a weak solution works also in non-symmetric domain once a solenoidal boundary extension satisfying the Leray-Hopf inequalities is given. In other words, if the flux over each connected inner boundary is sufficiently small, one can using the same technique to show the existence result.*

Theorem 2.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is an admissible domain given in Definition 2.5. Assume that the boundary value $\boldsymbol{a}$ is a symmetric field in $\mathbb{W}^{1/2,2}(\partial\Omega)$ having a compact support, the external force $\mathbf{f}$ is a symmetric vector field such that for every k the integral $\int\limits_{\Omega_k} \mathbf{f} \cdot \boldsymbol{\eta}\, dx$ defines a bounded functional on $\mathbb{H}(\Omega_k)$. Then problem (2.2) admits at least one weak solution $\mathbf{u} = \mathbf{A} + \mathbf{v}$. Furthermore, if $\int\limits_{R_{j,0}}^{+\infty} \frac{dx_1}{g_j^3(z_1^{(j)})} < +\infty$, $j = 1, ..., N$, and $z_1^{(j)}$ denotes the local coordinate system in outlet j, then the weak solution $\mathbf{u}$ satisfies the estimate*

$$\int\limits_{\Omega} |\nabla \boldsymbol{u}|^2\, dx \le c(\mathbf{a}, \|\mathbf{f}\|_*) \left(1 + \sum_{j=1}^{N} \int\limits_{R_{j,0}}^{+\infty} \frac{dz_1^{(j)}}{g_j^3(z_1^{(j)})}\right) \tag{2.12}$$

while if there is a number $j \in \{1, ..., N\}$ such that

$$\int\limits_{R_{j,0}}^{+\infty} \frac{dz_1^{(j)}}{g_j^3(z_1^{(j)})} = +\infty, \text{ and } \frac{\int\limits_{R_{j,k-1}}^{R_{j,k}} \frac{dx_1}{g^3(z_1^{(j)})}}{1 + \int\limits_{R_{j,0}}^{R_{j,k}} \frac{dz_1^{(j)}}{g^3(z_1^{(j)})}} \to 0 \text{ when } k \to +\infty, \text{ then } \mathbf{u} \text{ satisfies}$$

$$\int\limits_{\Omega_k} |\nabla \boldsymbol{u}|^2\, dx \le c(\mathbf{a}, \|\mathbf{f}\|_*) \left(1 + \sum_{j=1}^{N} \int\limits_{R_{j,0}}^{R_{j,k}} \frac{dz_1^{(j)}}{g_j^3(z_1^{(j)})}\right), \tag{2.13}$$

where

$$\|\mathbf{f}\|_* = \sup_{k \ge 1} \left(\Big(1 + \sum_{j=1}^{N} \int\limits_{R_{j,0}}^{R_{j,k}} \frac{dz_1^{(j)}}{g_j^3(z_1^{(j)})}\Big)^{-1/2} \cdot \|\mathbf{f}\|_{H^*(\Omega_k)} \right),$$

$$\|\mathbf{f}\|_{H^*(\Omega_k)} = \sup_{\boldsymbol{\eta} \in J_0^\infty(\Omega_k)} \frac{\Big|\int\limits_{\Omega_k} \mathbf{f} \cdot \boldsymbol{\eta}\, dx\Big|}{\|\boldsymbol{\eta}\|_{D(\Omega_k)}},$$

where $D(\Omega)$ is the closure of $\mathbb{C}_0^\infty(\Omega)$ w.r.t the Dirichlet norm, $c(\mathbf{a}, \|\mathbf{f}\|_*) = c\bigg(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{f}\|_*^2 \bigg)$ and c is independent of k.

We prove the existence theorem at the end of this chapter. We show first the results in a bounded domain. Then in the following few sections we construct a suitable extensions $\mathbf{A}$ of the boundary value $\mathbf{a}$ in each type of domains.

3 The Boundary Value Problem in Bounded Domains

As we explained in the previous section, in order to solve the Navier-Stokes equations in an unbounded domain Ω we exhaust our domain by a sequence of bounded domains Ω_l. We show the existence of a solution $\mathbf{u}_l$ in each Ω_l. Then we pass to the limit and obtain the desired solution. Firstly, let us consider the Navier-Stoke equations

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{a} \quad \text{on} \quad \partial\Omega \end{cases} \tag{3.1}$$

in a bounded domain Ω.

Definition 3.7. *We call* $\mathbf{u}$ *a weak solution to problem* (3.1) *if*

(i) $\mathbf{u}\in\mathbb{H}(\Omega)$ *satisfies*

$$\nu\int\limits_\Omega \nabla\mathbf{u}:\nabla\boldsymbol{\eta}\,dx + \int\limits_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\eta}\,dx = \int\limits_\Omega \mathbf{f}\cdot\boldsymbol{\eta},\ \forall\boldsymbol{\eta}\in\mathbb{J}_0^\infty(\Omega).$$

(ii) $\mathbf{u}=\mathbf{a}$ *on* $\partial\Omega$.

The Homogeneous Boundary Value Problem

We solve in this section firstly the stationary Navier–Stokes equations with homogenous boundary condition in a bounded domain Ω.

$$\begin{cases} -\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \\ \mathbf{u} = 0 \text{ in } \partial\Omega. \end{cases} \tag{3.2}$$

The existence of a weak solution was established by Ladyzhenskaya in [33]. We formulate the theorem and give a sketch of the proof.

Theorem 3.2. *Let* Ω *be a bounded locally Lipschitz domain of* $\mathbb{R}^2$ *with* $\partial\Omega$ *composed of* $m+1$ *connected components* $\Gamma_0,\dots,\Gamma_m$, $m\ge 1$. *Assume that* $\int_\Omega \mathbf{f}\cdot\boldsymbol{\eta}\,dx$ *defines a linear functional in* $\mathbb{H}(\Omega)$. *Then there exists a weak solution* $\mathbf{u}$ *to problem* (3.2).

Proof. By the definition 3.7 we look for a vector field $\mathbf{u}$ satisfying the following the integral identity:

$$\nu\int\limits_\Omega \nabla\mathbf{u}:\nabla\boldsymbol{\eta}\,dx - \int\limits_\Omega(\mathbf{u}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}\,dx = \int\limits_\Omega \mathbf{f}\cdot\boldsymbol{\eta}\,dx, \qquad \forall\boldsymbol{\eta}\in\mathbb{J}_0^\infty(\Omega). \tag{3.3}$$

According to the Riesz theorem there exists a unique element $\mathbf{L} \in \mathbb{H}(\Omega)$ such that the functional $\int\limits_\Omega \mathbf{f} \cdot \boldsymbol{\eta}$ can be represented by the form

$$\int\limits_\Omega \mathbf{f} \cdot \boldsymbol{\eta} = (\nabla \mathbf{L}, \nabla \boldsymbol{\eta}).$$

In fact, the integral $\int\limits_\Omega (\mathbf{u} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{u}\, dx$ defines also a linear functional of $\boldsymbol{\eta}$ for a fixed $\mathbf{u} \in \mathbb{H}(\Omega)$, where $\mathbb{H}(\Omega)$ denotes the closure of $\mathbb{J}_0^\infty(\Omega)$ with respect to the Dirichlet norm.

Note that since

$$\begin{aligned}|((\mathbf{u} \cdot \nabla)\boldsymbol{\eta}, \mathbf{u})| &\leq c\|\mathbf{u}\|_{L^4}^2 \|\nabla \boldsymbol{\eta}\| \\ &\leq C\|\nabla \mathbf{u}\|^2 \|\nabla \boldsymbol{\eta}\|,\end{aligned}$$

holds, the functional defined by $\int\limits_\Omega (\mathbf{u} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{u}\, dx$ is bounded. Thus, there exists a unique element $\mathcal{A}\mathbf{u} \in \mathbb{H}(\Omega)$ such that

$$\int\limits_\Omega (\mathbf{u} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{u}\, dx = (\nabla(\mathcal{A}\mathbf{u}), \nabla \boldsymbol{\eta}), \tag{3.4}$$

Hence, we can represent the integral identity (3.3) by the following equation

$$(\nabla(\nu \mathbf{u} - \mathcal{A}\mathbf{u} - L), \nabla \boldsymbol{\eta}) = 0, \forall \boldsymbol{\eta} \in \mathbb{J}_0^\infty(\Omega).$$

We therefore reduced the problem of finding a solution of (3.2) to solve the problem whether there exists a $\mathbf{u} \in \mathbb{H}(\Omega)$ satisfying the operator equation

$$\nu \mathbf{u} - \mathcal{A}\mathbf{u} - L = 0. \tag{3.5}$$

Now we want to apply the Leray-Schauder theorem to the operator equation

$$\frac{1}{\nu}(\mathcal{A}\mathbf{u} + L) = \mathbf{u}.$$

We need to show that $\mathcal{A}$ is a completely continuous operator of $\mathbb{H}(\Omega)$, and all possible solutions of above equation are uniformly bounded.

To see that $\mathcal{A}$ is completely continuous, let $\{\mathbf{u}_n\}$ be a weakly convergent sequence in $\mathbb{H}(\Omega)$, then $\{\mathbf{u}_n\}$ converges strongly in $\mathbb{L}^4(\Omega)$ to the limit $\mathbf{u}$. To see that $\mathcal{A}\mathbf{u}_n$ converges strongly in $\mathbb{H}(\Omega)$, use (3.4) we have

$$\begin{aligned}&(\nabla(\mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n), \nabla \boldsymbol{\eta}) \\ &= \textstyle\int_\Omega ((\mathbf{u}_m - \mathbf{u}_n) \cdot \nabla)(\mathbf{u}_m - \mathbf{u}_n) \cdot \boldsymbol{\eta} dx \\ &= \textstyle\int_\Omega ((\mathbf{u}_m - \mathbf{u}_n) \cdot \nabla)\mathbf{u}_m \cdot \boldsymbol{\eta} dx + \int_\Omega (\mathbf{u}_n \cdot \nabla)(\mathbf{u}_m - \mathbf{u}_n) \cdot \boldsymbol{\eta} dx \\ &\leq C\|\mathbf{u}_m - \mathbf{u}_n\|_{L^4(\Omega)}(\|\mathbf{u}_m\|_H + \|\mathbf{u}_n\|_H)(\|\boldsymbol{\eta}\|_H),\end{aligned} \tag{3.6}$$

where we used the Hölder inequality and the Poincaré inequality for the last estimate. Set $\boldsymbol{\eta} = \mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n$ into the above inequality and use the fact that the norm of $\{\mathbf{u}_n\}$ is bounded we derive that

$$\|\nabla(\mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n)\| \leq C\|\mathbf{u}_m - \mathbf{u}_n\|_{L^4(\Omega)}, \tag{3.7}$$

which converges to zero as m, n go to infinity. It is left to show that all the possible solutions to (3.5) are uniformly bounded. Take the scalar product with $\mathbf{u}$

$$(\nabla(\mathbf{u} - \frac{1}{\nu}(\mathcal{A}\mathbf{u} + L)), \nabla\mathbf{u}) = 0. \tag{3.8}$$

It follows

$$\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{u} - \lambda(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}dx = \lambda\int_\Omega \mathbf{f}\cdot\mathbf{u}dx, \tag{3.9}$$

where $\lambda \in [0, \frac{1}{\nu}]$. Since the nonlinear term vanishes, we derive the uniform bound of $\|\mathbf{u}\|_H$ that

$$\|\mathbf{u}\|_H \leq \lambda\|L\|_H = \lambda|\mathbf{f}|. \tag{3.10}$$

□

Remark 3.6. *This proof uses the Leray Schauder fixed point theorem, which gives only an existence result. If no further restriction is given we cannot show the uniqueness of the solution.*

The nonhomogeneous Boundary Value Problem

For the nonhomogeneous case we are looking for a solution of the form $\mathbf{u} = \mathbf{v} + \mathbf{A}$, where $\mathbf{A}$ is a solenoidal extension of the boundary value $\mathbf{a}$ into Ω. From (3.1) we derive that $\mathbf{v}$ solves the following problem.

$$\begin{aligned}&\nu\int_\Omega \nabla\mathbf{v} : \nabla\boldsymbol{\eta}\,dx - \int_\Omega((\mathbf{A}+\mathbf{v})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{v}\,dx - \int_\Omega(\mathbf{v}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{A}\,dx\\ &= \int_\Omega(\mathbf{A}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{A}\,dx - \nu\int_\Omega\nabla\mathbf{A} : \nabla\boldsymbol{\eta}\,dx + \int_\Omega\mathbf{f}\cdot\boldsymbol{\eta}\,dx, \qquad \forall\boldsymbol{\eta}\in\mathbb{J}_0^\infty(\Omega).\end{aligned} \tag{3.11}$$

In order to show the existence by the same technique as used for the homogeneous case we need to find a uniform bound of $\mathbf{v}$ in $\mathbb{W}^{1,2}$. If we set $\boldsymbol{\eta} = \mathbf{v}$ into the above equation and integrating by part and applying Lemma 1.2 we derive

$$\begin{aligned}&\nu\int_\Omega\nabla\mathbf{v} : \nabla\mathbf{v}\,dx\\ &=\\ &\int_\Omega(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{A}\,dx + \int_\Omega(\mathbf{A}\cdot\nabla)\mathbf{v}\cdot\mathbf{A}\,dx - \nu\int_\Omega\nabla\mathbf{A} : \nabla\mathbf{v}\,dx + \int_\Omega\mathbf{f}\cdot\mathbf{v}\,dx.\end{aligned} \tag{3.12}$$

We can bound the last three terms on the right hand side of the above equation easily by $c\left(\int_\Omega \nabla\mathbf{v} : \nabla\mathbf{v}\right)^{1/2}$, with c a constant depending on Ω, $\mathbf{f}$ and $\mathbf{A}$. For the first term on the right hand side if the extension we constructed satisfies

$$\int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{A}\,dx \le \tilde{c}\|\mathbf{v}\|^2_{\mathbb{H}(\Omega)} \tag{3.13}$$

for a positive constant $\tilde{c} < \nu$ for all $\mathbf{v} \in \mathbb{H}(\Omega)$, then we obtain a uniform bound of $\mathbf{v}$ in $\mathbb{H}(\Omega)$. If we do not want to impose any restriction to the viscosity constant we need to find for any $\tilde{c}$ there exists a solenoidal extension $\mathbf{A}(\tilde{c})$ of the boundary value $\mathbf{a}$ such that the above estimate exists. This inequality is so called Leray-Hopf's inequality.

Another important point is that if we do not require that the extension to be solenoidal we could just extend the boundary value $\mathbf{a}$ to Ω and cut-off by a smooth function ψ_ε, ε indicates the support of the extension in Ω. In this way we can show that the Leray-Hopf inequality is satisfied. However, in the studying of the Navier-Stokes equations we need to extend the boundary value to a divergence free function. Therefore, we use the technique developed by Leray [37] and Hopf [18]. This technique works if the flux over each connected boundary equals to zero or sufficiently small (see for example [16]). We prove the following lemma.

Lemma 3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$, for simplicity we assume that $\partial\Omega$ contains only one connected inner boundary Γ and $\mathbf{a} \in \mathbb{W}^{1/2,2}(\partial\Omega)$ satisfies*

$$\int_\Gamma \mathbf{a}\cdot\mathbf{n}\,dx = 0.$$

Then, for each $\varepsilon > 0$ there exists a solenoidal vector field $\mathbf{A} \in \mathbb{W}^{1,2}(\Omega)$ such that $\mathbf{A}|_{\partial\Omega} = \mathbf{a}$ and satisfying

$$\int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{A}\,dx \le \varepsilon\|\mathbf{v}\|^2_{\mathbb{H}(\Omega)}\ \forall \mathbf{v} \in \mathbb{H}(\Omega). \tag{3.14}$$

Proof. Since

$$\int_\Gamma \mathbf{a}\cdot\mathbf{n}\,dx = 0$$

holds, by Remark 1.1 we know that there exists an solenoidal extension $\tilde{\mathbf{a}} = (\tilde{a}_1, \tilde{a}_2) \in \mathbb{W}^{1,2}(\Omega)$ of $\mathbf{a}$ satisfying

$$\|\tilde{\mathbf{a}}\|_{W^{1,2}} \le C\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}.$$

We choose then a fixed point $p \in \Omega$ and define a function

$$l(x) := \int_p^x \tilde{a}_1 dx_2 - \tilde{a}_2 dx_1,\ x \in \Omega.$$

Because of the flux on each connected boundary equals to zero, this function is well defined. Clearly, it holds that

$$\frac{\partial l}{\partial x_2} = \tilde{a}_1$$

and

$$\frac{\partial l}{\partial x_1} = -\tilde{a}_2.$$

Now we want to construct a suitable cut-off function ψ_ε so that our extension has the form of $\nabla \times (\psi_\varepsilon l)(x) := (\partial_2(\psi_\varepsilon l)(x), -partial_1(\psi_\varepsilon l)(x))$.

Let $\delta_0,\ \kappa_0 \in \mathbb{R}$ such that $\delta_0 > 0,\ 1/4 > \kappa_0 > 0$. Let $j(t) \in C_0^\infty[0,\infty)$ be a function with following properties:

$$\begin{aligned}
&0 \le j(t) \le 1/t,\\
&j(t) = 0, \text{ for } 0 \le t \le \kappa_0\delta_0,\ (1-\kappa_0)\delta_0 \le t,\\
&j(t) = 1/t, \text{ for } 2\kappa_0\delta_0 \le t \le (1-2\kappa_0)\delta_0.
\end{aligned}$$

Define a function $h(t) = 1 - \frac{\int_0^t j(s)ds}{\int_0^\infty j(s)ds}$. Then

$$\begin{cases} 0 \le h(t) \le 1, & 0 \le t,\\ h(t) \equiv 1, & 0 \le t \le \kappa_0\delta_0,\\ h(t) \equiv 0, & (1-\kappa_0)\delta_0 \le t. \end{cases} \tag{3.15}$$

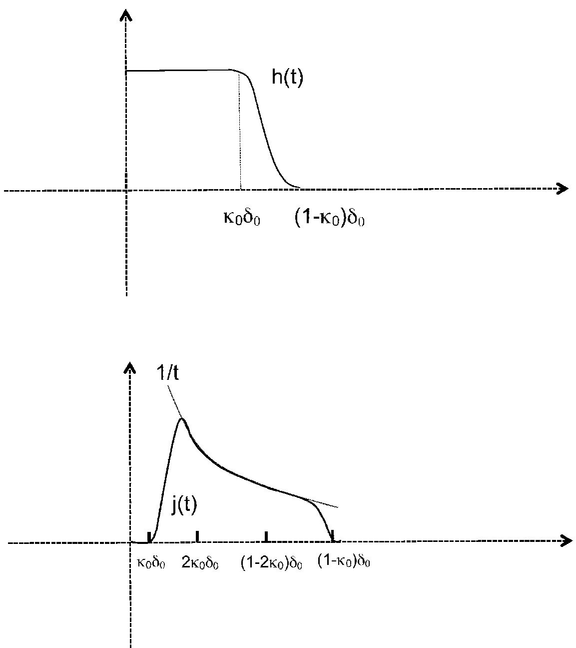


Figure 2.1: h(t) and j(t)

Since

$$
\begin{aligned}
&|t \cdot h'(t)| \\
&= \left| \frac{j(t)\cdot t}{\int_0^\infty j(s)ds} \right| \\
&\leq \left| \frac{1}{\int_{2\kappa_0\delta_0}^{(1-2\kappa_0)\delta_0} \frac{1}{s}} \right| \\
&= (log(1-2\kappa_0)\delta_0 - log(2\kappa_0\delta_0))^{-1} \\
&= \left(log\frac{1-2\kappa_0}{2\kappa_0} \right)^{-1}
\end{aligned}
\tag{3.16}
$$

then

$$
\sup_{0\leq t\leq\delta_0} |t\cdot h'(t)| \leq \left(log\frac{1-2\kappa_0}{2\kappa_0} \right)^{-1} \leq \{\underbrace{log(1-2\kappa_0)}_{\to 0 \text{ as } \kappa_0\to 0} - \underbrace{log 2\kappa_0}_{\to\infty \text{ as } \kappa_0\to 0}\}^{-1} \to 0 \text{ as } \kappa_0 \to 0.
\tag{3.17}
$$

Let $d(x)$ be the distance function to $\partial\Omega$, $\Delta(x)$ be the regularized distance function to $\partial\Omega$ for $x \in \Omega$ and define

$$
\mathbf{A}(x) = \nabla \times (h(\Delta(x))l(x)) \quad \text{for } x \in \Omega.
$$

From the properties of the regularized distance function we know that:

$$
\begin{aligned}
&d(x) \leq C\Delta(x); \\
&|\partial^{|k|}\Delta(x)| \leq \alpha_{|k|+1} d(x)^{1-|k|},
\end{aligned}
\tag{3.18}
$$

where $|k| = k_1 + k_2$.

For $x \in \partial\Omega$ we see that $\mathbf{A}(x) = \nabla \times (h(0)l(x)) = \nabla \times l(x) = \mathbf{a}(x)$ and it holds that

$$
\mathbf{A} = h(\Delta(x))\nabla \times l(x) + l(x)h'(\Delta(x))\nabla \times \Delta(x).
\tag{3.19}
$$

Define $\Omega_{\delta_0} := \{x \in \Omega \,|\, \Delta(x) < \delta_0\}$. Then, the support of $\mathbf{A}$ is contained in Ω_{δ_0}. Let $\mathbf{v} \in H^S(\Omega)$, by using the formula of lemma (1.1) $(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{2}\nabla|\mathbf{v}|^2 - \omega\mathbf{v}^\perp = \frac{1}{2}\nabla|\mathbf{v}|^2 - (D_1v_2 - D_2v_1)(v_2, -v_1)$, and note that $\operatorname{div}\mathbf{A} = 0$ and

$$
\int_\Omega \frac{1}{2}\nabla|\mathbf{v}|^2 \cdot \mathbf{A} = \frac{1}{2}\int_\Omega \operatorname{div}(|\mathbf{v}|^2\mathbf{A}) - \underbrace{\frac{1}{2}\int_\Omega |\mathbf{v}|^2 \operatorname{div}\mathbf{A}}_{=0} = \frac{1}{2}\int_{\partial\Omega} |\mathbf{v}|^2\mathbf{A}\cdot\mathbf{n}d\sigma = 0,
$$

we obtain

$$
|((\mathbf{v}\cdot\nabla)\mathbf{v}, \mathbf{A})| = |(\omega\mathbf{v}^\perp, \mathbf{A})| \leq \int_\Omega |\omega||\mathbf{v}^\perp||\mathbf{A}|dx \leq \|\nabla\mathbf{v}\|\||\mathbf{v}||\mathbf{A}|\|.
$$

Since $\operatorname{supp}\mathbf{A} \subset \Omega_{\delta_0}$, we obtain $\||\mathbf{v}||\mathbf{A}|\|^2 = \int_\Omega |\mathbf{v}|^2|\mathbf{A}|^2dx = \int_{\Omega_{\delta_0}} |\mathbf{v}|^2|\mathbf{A}|^2dx$. Using (3.19), we have

$$
|\mathbf{v}||\mathbf{A}| \leq |\mathbf{v}||h(\Delta(x))||\nabla \times l(x)| + \left|\frac{\mathbf{v}(x)}{\Delta(x)}\right| \Delta(x)|h'(\Delta(x))||\nabla\Delta||l(x)|.
\tag{3.20}
$$

Thus

$$\||\mathbf{v}||\mathbf{A}|\|_{L^2(\Omega_{\delta_0})} \leq \||\mathbf{v}||\nabla \times l|\|_{L^2(\Omega_{\delta_0})} + \left\|\frac{|\mathbf{v}|}{d(x)}\Delta h'(\Delta) l |\nabla \Delta|\right\|_{L^2(\Omega_{\delta_0})}. \tag{3.21}$$

We obtain the following estimate by using Hölder's inequality, Sobolev's embedding theorem and Poincaré's inequality.

$$\begin{aligned}
\||\mathbf{v}||\nabla \times l|\|_{L^2(\Omega_{\delta_0})} &\leq \||\mathbf{v}|\|_{L^4(\Omega_{\delta_0})}\||\nabla \times l|\|_{L^4(\Omega_{\delta_0})} \\
&\leq C_2 \||\nabla \mathbf{v}|\|_{L^2(\Omega_{\delta_0})}\||\nabla \times l|\|_{L^4(\Omega_{\delta_0})} \\
&\leq C_2 \||\nabla \mathbf{v}|\|_{L^2(\Omega)}\||\nabla \times l|\|_{L^4(\Omega_{\delta_0})}
\end{aligned}$$

Choose δ_0 sufficiently small, we can find a ε such that $\||\nabla \times l|\|_{L^4(\Omega_{\delta_0})} \leq \frac{1}{2C_2}\varepsilon$. We therefore obtain

$$\||\mathbf{v}||\nabla \times l|\|_{L^2(\Omega_{\delta_0})} \leq \frac{1}{2}\varepsilon \||\nabla \mathbf{v}|\|_{L^2(\Omega)}.$$

We estimate the second term of (3.21) as follows

$$\begin{aligned}
\left\|\frac{|\mathbf{v}|}{d(x)}\Delta h'(\Delta) l |\nabla \Delta|\right\|_{L^2(\Omega_{\delta_0})} &\leq C_3 \sup_{0\leq \Delta \leq \delta_0} |\Delta h'(\Delta)| \|l\|_{L^\infty(\Omega_{\delta_0})} \left\|\frac{|\mathbf{v}|}{\Delta}\right\|_{L^2(\Omega_{\delta_0})} \\
&\leq C_4 \sup_{0\leq \Delta \leq \delta_0} |\Delta h'(\Delta)| \|l\|_{L^\infty(\Omega_{\delta_0})} \|\nabla \mathbf{v}\|.
\end{aligned}$$

For the second estimate of the above inequality we applied the Hardy inequality. Using then the estimate of (3.17) and choosing κ_0 sufficiently small we obtain

$$C_4 \sup_{0\leq \Delta \leq \delta_0} |\Delta h'(\Delta)| \|l\|_{L^\infty(\Omega_{\delta_0})} \|\nabla \mathbf{v}\| \leq \frac{1}{2}\varepsilon \|\nabla \mathbf{v}\|.$$

Therefore, (LI) can be estimated as

$$|((\mathbf{v}\cdot\nabla)\mathbf{v}, \mathbf{A})| \leq \|\nabla \mathbf{v}\| \||\mathbf{v}||\mathbf{A}|\| = \|\nabla \mathbf{v}\| \||\mathbf{v}||\mathbf{A}|\|_{L^2(\Omega_{\delta_0})} \leq \varepsilon \|\nabla \mathbf{v}\|^2.$$

□

4 Construction of the Extension

We introduce now the construction of an extension **A** of the boundary value **a**. The construction can be considered as two parts. For the inner boundary if the flux over each connected component is equal to zero one can apply Lemma 3.1 to show that there exists a solenoidal extension satisfying the Leray-Hopf inequality. Notice that the symmetry assumptions are not needed in this case. If the flux over the each connected component of the inner boundary is arbitrary large we can only construct a solenoidal extension under certain symmetry assumptions. For a suitable extension of the outer boundary value the symmetry assumptions are not necessary. We start

with the construction of the boundary value extension in a general two dimensional domain.
Next we construct a suitable extension of the given boundary value having arbitrary large flux in symmetric domains.
In the last part of this chapter we show that there exists at least one solution once we have a suitable boundary value extension. Note that the technique used for showing the existence result works in general domains. We start with introducing some auxiliary functions which are introduced by V. A. Solonnikov and improved by M. Chipot.

Solenoidal vector fields carrying a constant flux to infinity

Let γ, $\overline{\gamma}$ be two smooth curves. We consider the following two cases:

(i) If γ intersects one boundary of D_i, we define then $\overline{\gamma}$ to be an infinite part of ∂D_i such that dist $(\gamma, \overline{\gamma}) > 0$. (see the upper part of the figure below)

(ii) If γ does not intersect any outer boundaries, $\overline{\gamma}$ is then defined to be one outer boundary Γ_0^i (see the lower part of the figure below).

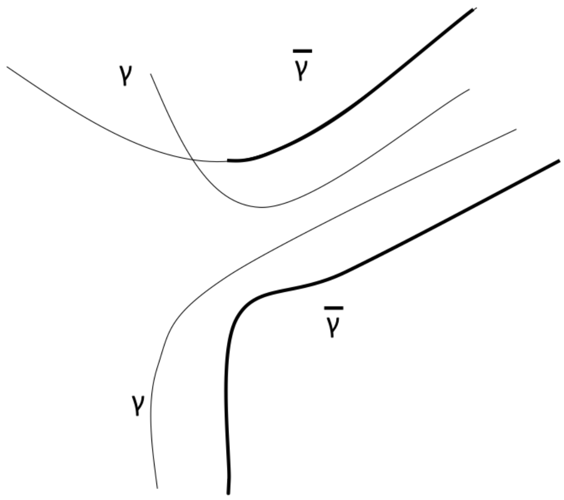


Figure 2.2: Smooth curves γ and $\overline{\gamma}$

We use γ intersecting the outer boundary to construct the extension $\mathbf{B}^{(out)}$ and use γ not intersecting any outer boundaries to construct $\mathbf{B}^{(flux)}$.
Let Ψ be a smooth cut of function such that

$$\Psi(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t \geq 1. \end{cases} \tag{4.1}$$

Then define for x restricted between γ and $\overline{\gamma}$ a function ξ by the formula

$$\xi(x,\varepsilon) = \Psi\Big(\varepsilon \ln \frac{\Delta_\gamma(x)}{\Delta_{\overline{\gamma}}}\Big), \tag{4.2}$$

where $\Delta_M(x)$ denotes the regularized distance function to a set M.

Lemma 4.1. *The function $\xi(x)$ is equal to zero for $\Delta_\gamma \le \Delta_{\overline{\gamma}}$, $\xi = 1$ for $\Delta_\gamma \ge e^{1/\varepsilon}\Delta_{\overline{\gamma}}$. In particular $\xi = 0$ in a neighborhood of γ, $\xi = 1$ in a neighborhood of $\overline{\gamma}$, and the support of ξ is contained in the set where*

$$e^{-1/\varepsilon}\Delta_\gamma \le \Delta_{\overline{\gamma}} \le \Delta\gamma. \tag{4.3}$$

Furthermore, the following inequalities hold for $x \in$supp ξ.

$$\Big|\frac{\partial \xi(x,\varepsilon)}{\partial x_k}\Big| \le \frac{c\,\varepsilon}{\Delta_\gamma(x)}, \qquad \Big|\frac{\partial^2 \xi(x,\varepsilon)}{\partial x_k \partial x_l}\Big| \le \frac{c\,\varepsilon}{\Delta_\gamma^2(x)} \tag{4.4}$$

$$\Big|\frac{\partial \xi(x,\varepsilon)}{\partial x_k}\Big| \le \frac{c(\varepsilon)}{g(x_1)}, \qquad \Big|\frac{\partial^2 \xi(x,\varepsilon)}{\partial x_k \partial x_l}\Big| \le \frac{c(\varepsilon)}{g^2(x_1)} \tag{4.5}$$

Proof. We prove the inequalities (4.4). Let $x \in$supp ξ

$$\begin{aligned} &\Big|\frac{\partial \xi(x,\varepsilon)}{\partial x_k}\Big| \le \Big|\Psi'\varepsilon\Big(\frac{\partial x_k \Delta_\gamma}{\Delta_\gamma} - \frac{\partial x_k \Delta_{\overline{\gamma}}}{\Delta_{\overline{\gamma}}}\Big)\Big| \\ &\le c\varepsilon\Big(\frac{1}{\Delta\gamma} + \frac{1}{\Delta_{\overline{\gamma}}}\Big) \le \frac{c\varepsilon}{\Delta_{\overline{\gamma}}} \le \frac{c\varepsilon}{d(x,\overline{\gamma})}. \end{aligned} \tag{4.6}$$

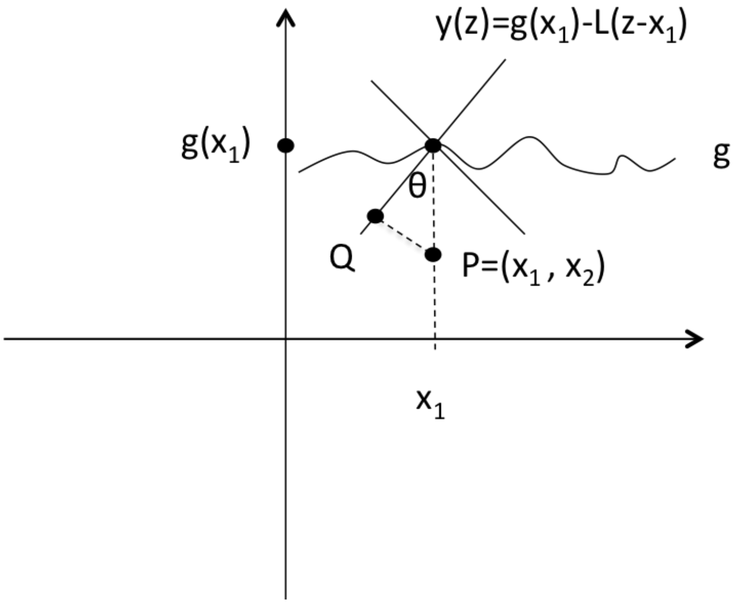
We used the properties of the regularized distance function, $\partial_{x_k}\Delta_\gamma$ and $\partial_{x_k}\Delta_{\overline{\gamma}}$ are uniformly bounded, (4.3) and $\Delta_{\overline{\gamma}} \ge cd(x,\overline{\gamma})$ for some constant. Computing the second derivatives one has

$$\begin{aligned} &\frac{\partial^2 \xi(x,\varepsilon)}{\partial x_k \partial x_l} = \Psi''\varepsilon^2\Big(\frac{\partial_{x_l}\Delta_\gamma}{\Delta_\gamma} - \frac{\partial_{x_l}\Delta_{\overline{\gamma}}}{\Delta_{\overline{\gamma}}}\Big)\Big(\frac{\partial_{x_k}\Delta_\gamma}{\Delta_\gamma} - \frac{\partial_{x_k}\Delta_{\overline{\gamma}}}{\Delta_{\overline{\gamma}}}\Big) \\ &+\Psi'\varepsilon\Big(-\frac{\partial_{x_l}\Delta_\gamma \partial_{x_k}\Delta_\gamma}{\Delta_\gamma^2} + \frac{\partial^2_{x_k x_l}\Delta_\gamma}{\Delta_\gamma} + \frac{\partial_{x_l}\Delta_{\overline{\gamma}}\partial_{x_k}\Delta_{\overline{\gamma}}}{\Delta_{\overline{\gamma}}^2} - \frac{\partial^2_{x_k x_l}\Delta_{\overline{\gamma}}}{\Delta_{\overline{\gamma}}}\Big). \end{aligned} \tag{4.7}$$

Using the same arguments for (4.6) one has $\Big|\partial^2_{x_k x_l}\Delta\Big| \le \dfrac{C}{\Delta}$ for some constant C and

$$\begin{aligned} &\Big|\frac{\partial^2 \xi(x,\varepsilon)}{\partial x_k \partial x_l}\Big| \le C\varepsilon\Big(\big(\frac{1}{\Delta_\gamma} + \frac{1}{\Delta_{\overline{\gamma}}}\big)^2 + \big(\frac{1}{\Delta_\gamma^2} + \frac{1}{\Delta_{\overline{\gamma}}^2}\big)\Big) \\ &\le \frac{C\varepsilon}{\Delta_{\overline{\gamma}}^2} \le \frac{C\varepsilon}{d(x,\overline{\gamma})}. \end{aligned} \tag{4.8}$$

Note that the constants C in (4.6),(4.8) are independent of ε. Let us prove the inequalities (4.5). We say that a quantity $A \sim B$ (A is equivalent to B) when there exists positive constants α, β, such that $\alpha B \le A \le \beta B$, clearly $B \sim A$. Since the



regularized distance function is equivalent to the usual distance function denoted by $d(x, M)$ for a set M, from (4.3) it follows that in the support of ξ one has

$$\Delta_\gamma \sim \Delta_{\overline{\gamma}} \sim d(x, \overline{\gamma}) \sim d(x, \gamma). \tag{4.9}$$

with the constants depending possibly on ε. We assume further that the smooth curve γ coincides with the local x_1-axis for x_1 large enough. Note that due to the Lipschitz condition of g the graph of g is bounded between the straight lines going through $(x_1, g(x_1))$ with slopes $\pm L$ (see the graphic). One has then with the notation of the figure, θ depending on L only

$$\begin{aligned} g(x_1) &= x_2 + g(x_1) - x_2 \geq x_2 = d(x, \gamma), \quad x = (x_1, x_2), \\ g(x_1) &= x_2 + g(x_1) - x_2 = d(x, \gamma) + g(x_1) - x_2 \\ &= d(x, \gamma) + \frac{PQ}{\sin(\theta)} \leq d(x, \gamma) + \frac{d(x, \overline{\gamma})}{\sin(\theta)}. \end{aligned} \tag{4.10}$$

It follows from (4.9) that in the support of ξ $g(x_1) \sim d(x, \overline{\gamma})$ and for some constant $c(\varepsilon)$ one has

$$d(x, \overline{\gamma}) \geq c(\varepsilon) g(x_1)$$

which gives for the same arguments of (4.6) and (4.8) the following estimates

$$\Big|\frac{\partial \xi(x, \varepsilon)}{\partial x_k}\Big| \leq \frac{C(\varepsilon)}{g(x_1)}, \qquad \Big|\frac{\partial^2 \xi(x, \varepsilon)}{\partial x_k \partial x_l}\Big| \leq \frac{C(\varepsilon)}{g^2(x_1)}. \tag{4.11}$$

□

4.1 General Domains with N outlets

We consider in this section the general domain containing finitely many outlets. We use the solenoidal vector fields carrying a constant flux to infinity (4.2) introduced in the previous section. These vector fields help to drain the fluxes from each outer boundary to infinity and the flux coming from each outlet to the next outlet. In this way we can reduce the problem to the case of zero flux over each connected boundary. Furthermore, we assume in this section that the flux over each connected inner boundary to be zero, therefore as mentioned before we do not need to assume the domain to be symmetric. Note that this section is due to M. Chipot.

Lemma 4.2. *Let Ω be defined as in Section 2.1 without symmetric assumption and Ω_{k+1}, ω_k be defined as on page 15. Let $\mathbf{a} \in \mathbb{W}^{1/2,2}(\partial\Omega)$ satisfying*

$$\begin{aligned} &\textstyle\int_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\,dS = 0, \quad i = 1,\ldots,I, \\ &\textstyle\int_{\Gamma_0^m} \mathbf{a}\cdot\mathbf{n}\,dS = \mathbb{F}_m^{(out)}, \quad m = 1,\ldots,N, \end{aligned} \tag{4.12}$$

Then there exists a solenoidal extension $\mathbf{A}$ of the $\mathbf{a}$ satisfying the Leray-Hopf inequalities, i.e., for every solenoidal function $\mathbf{w} \in \mathbb{W}^{1,2}_{loc}(\overline{\Omega})$ with $\mathbf{w}|_{\partial\Omega} = 0$ the following estimates

$$\begin{aligned} &\Big|\int\limits_{\Omega_{k+1}} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{A}\,dx\Big| \le c\,\varepsilon \int\limits_{\Omega_{k+1}} |\nabla\mathbf{w}|^2\,dx, \\ &\Big|\int\limits_{\omega_k} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{A}\,dx\Big| \le c\,\varepsilon \int\limits_{\omega_k} |\nabla\mathbf{w}|^2\,dx \end{aligned} \tag{4.13}$$

hold. The constant c is independent of k and ε. Moreover,

$$\begin{aligned} &|\mathbf{A}| \le \frac{C(\varepsilon)}{g_j^2(z_1^{(j)})}, \\ &|\nabla\mathbf{A}| \le \frac{C(\varepsilon)}{g_j^3(z_1^{(j)})}, \quad x \in D_j,\ j = 1,\ldots,N. \end{aligned} \tag{4.14}$$

and it holds

$$\int\limits_{\sigma_j(R)} \mathbf{A}\cdot\mathbf{n}\,dS = \mathbb{F}_j, \quad j = 1,\cdots,N,$$

where $\mathbf{n}$ denotes the outward unit normal vector to $\partial\Omega$ or to the cross section $\sigma_j(R)$.

Proof. We start with a domain containing $N \ge 2$ outlets. We number the outlets D_j, $j = 1,\ldots N$ clockwise as showed in Figure 2.3. Each D_j, $j = 1,\ldots,N$ is enclosed by two infinite outer boundaries Γ_0^j and Γ_0^{j+1}. D_N is enclosed by Γ_0^N and Γ_0^1.
On each outlet D_j define two smooth curves γ_j and $\gamma_{j,j+1}$ with γ_j intersecting the outer boundary Γ_0^j and $\gamma_{j,j+1}$ contained in D_j and D_{j+1}. As introduced before, $\overline{\gamma}_j$

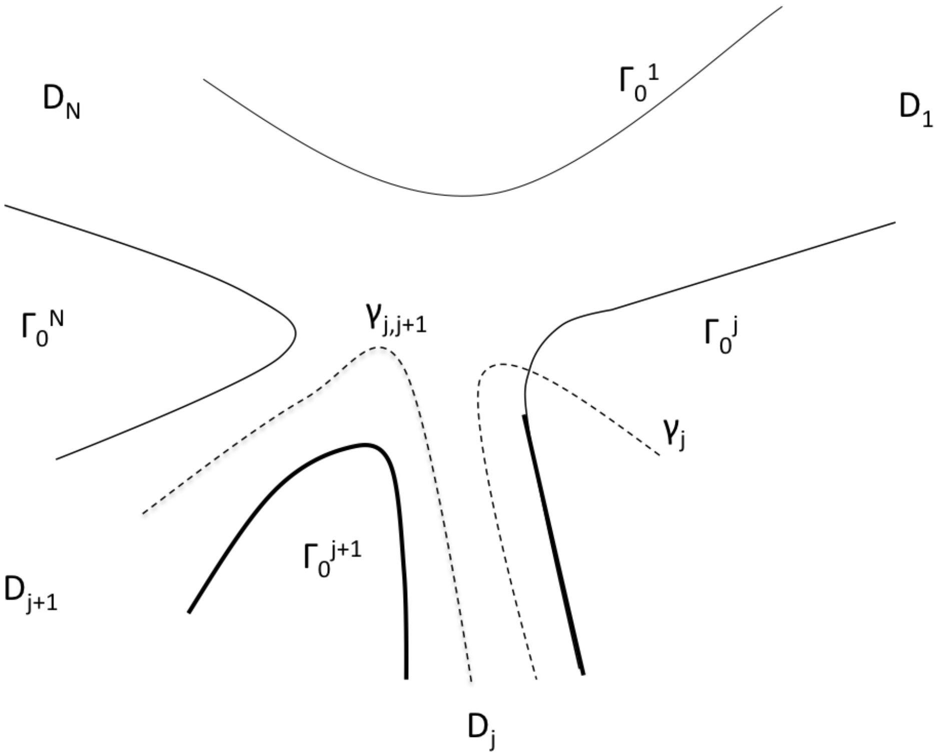


Figure 2.3: General domain with N outlets

denotes an infinite part of Γ_0^j and $\overline{\gamma}_{j,j+1}$ denotes Γ_0^{j+1} accordingly. Define further the following functions

$$\mathbf{b}_j = -(\frac{\partial \xi_j}{\partial x_2}, -\frac{\partial \xi_j}{\partial x_1})$$

and

$$\mathbf{d}_{j,j+1} = (\frac{\partial \xi_{j,j+1}}{\partial x_2}, -\frac{\partial \xi_{j,j+1}}{\partial x_1})$$

where $\xi_j(x,\varepsilon) = \Psi\left(\varepsilon \ln \dfrac{\Delta_{\gamma_j}(x)}{\Delta_{\overline{\gamma}_j}}\right)$ and $\xi_{j,j+1}(x,\varepsilon) = \Psi\left(\varepsilon \ln \dfrac{\Delta_{\gamma_j}(x)}{\Delta_{\overline{\gamma}_{j,j+1}}}\right)$ are defined as in Lemma 4.1.

Clearly $\mathbf{b}_j$ and $\mathbf{d}_{j,j+1}$ are solenoidal. Therefore, one has

$$\begin{aligned} &\int_{\Gamma_0^j} \mathbf{b}_j \cdot \mathbf{n} dS = -\int_{\sigma_j(R)} \mathbf{b}_j \cdot \mathbf{n} dS = 1, \\ &\int_{\sigma_j(R)} \mathbf{d}_{j,j+1} \cdot \mathbf{n} dS = -\int_{\sigma_{j+1}(R)} \mathbf{d}_{j,j+1} \cdot \mathbf{n} dS = -1. \end{aligned} \tag{4.15}$$

Moreover, on each outer boundary Γ_0^j and each inner boundary Γ_i one has

$$\begin{aligned} &\int_{\Gamma_0^j} (\mathbf{a} - \textstyle\sum_{j=1}^N \mathbb{F}_j^{(out)} \mathbf{b}_j) \cdot \mathbf{n}\, dS = 0. \\ &\int_{\Gamma_i} (\mathbf{a} - \textstyle\sum_{j=1}^N \mathbb{F}_j^{(out)} \mathbf{b}_j) \cdot \mathbf{n}\, dS = 0 \end{aligned} \tag{4.16}$$

Notice that the second equation holds because of the flux condition of the inner boundary value and the fact that the support of $\mathbf{b}_j$ does not intersect the inner boundary of Ω.

Hence because of Lemma 3.1 there exists a solenoidal extension $\mathbf{B}_0$ of the boundary value $\mathbf{a} - \sum_{j=1}^{N} \mathbb{F}_j^{(out)} \mathbf{b_j}$ satisfying the Leray-Hopf inequalities.

Set

$$\mathbf{A}_0 = \mathbf{B}_0 + \sum_{j=1}^{N} \mathbb{F}_j^{(out)} \mathbf{b_j},$$

then $\mathbf{A}_0$ is a solenoidal extension of the boundary value $\mathbf{a}$. It remains to show that $\mathbf{A}_0$ satisfies the Leray-Hopf inequalities. It is enough to show that each $\mathbf{b}_j$ satisfies the Leray-Hopf inequalities.

Let $\mathbf{w} = (w_1, w_2) \in \mathbb{W}^{1,2}_{loc}(\overline{\Omega})$, $\mathbf{w}|_{\partial\Omega} = 0$, be a solenoidal vector field. Applying the Hölder inequality, Lemma 4.1 and Lemma 1.12 we obtain:

$$\begin{aligned}
&|\textstyle\int_{\Omega_{k+1}} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{b}_j\,dx| \\
&\leq \Big(\textstyle\int_{\Omega_{k+1}} |\nabla \mathbf{w}|^2\,dx \Big)^{1/2} \cdot \Big(\textstyle\int_{\Omega_{k+1}} |\mathbf{w}\cdot\mathbf{b}|^2\,dx \Big)^{1/2} \\
&\leq \Big(\textstyle\int_{\Omega_{k+1}} |\nabla \mathbf{w}|^2\,dx \Big)^{1/2} \cdot \Big(\textstyle\int_{\Omega_{k+1}} |\mathbf{w}|^2 \cdot \Big(\dfrac{c\varepsilon}{d_{\overline{\gamma}}}\,dx \Big)^2 \Big)^{1/2} \\
&\leq c\varepsilon \textstyle\int_{\Omega_{k+1}} |\nabla \mathbf{w}|^2\,dx.
\end{aligned} \tag{4.17}$$

By using the same arguments we can prove the second inequality of (4.13).

Set then

$$\mathbf{A} = \mathbf{A}_0 + \textstyle\sum_{j=1}^{N-1} l_j \mathbf{d}_{j,j+1}, \tag{4.18}$$

with

$$l_j = \sum_{k=1}^{j} \big(\mathbb{F}_k + \mathbb{F}_k^{(out)} \big).$$

Since $\mathbf{d}_{j,j+1}$ vanishes on $\partial\Omega$ we have

$$\mathbf{A} = \mathbf{A}_0 = \mathbf{a} \quad on\ \partial\Omega.$$

Moreover, on an outlet D_j the only non vanishing functions of formula (4.18) are $\mathbf{b}_j$, $\mathbf{d}_{j-1,j}$ and $\mathbf{d}_{j,j+1}$. Hence,

$$\begin{aligned}
&\textstyle\int_{\sigma_j(R)} \mathbf{A}\cdot\mathbf{n}\,dS, \\
&= \textstyle\int_{\sigma_j(R)} \Big(\mathbb{F}_j^{(out)} \mathbf{b}_j + l_{j-1}\mathbf{d}_{j-1,j} + l_j \mathbf{d}_{j,j+1} \Big) \cdot \mathbf{n}\,dS.
\end{aligned} \tag{4.19}$$

Because of (4.15), the above integral is equal to

$$-\mathbb{F}_j^{(out)} - \sum_{k=1}^{j-1} \Big(\mathbb{F}_k + \mathbb{F}_k^{(out)} \Big) + \sum_{k=1}^{j} \Big(\mathbb{F}_k + \mathbb{F}_k^{(out)} \Big) = \mathbb{F}_j.$$

Note for $j = 1$ we only have $\mathbf{b}_1$, $\mathbf{d}_{1,2}$ are non vanishing on D_1. We have therefore

$$\begin{aligned}&\textstyle\int_{\sigma_1(R)} \mathbf{A}\cdot\mathbf{n}\,dS,\\ &= -\mathbb{F}_1^{(out)} + \mathbb{F}_1 + \mathbb{F}_1^{(out)} = \mathbb{F}_1.\end{aligned} \tag{4.20}$$

For $j = N$ we have non vanishing functions $\mathbf{b}_N$, $\mathbf{d}_{N-1,N}$ on D_N. Thus

$$\begin{aligned}&\textstyle\int_{\sigma_N(R)} \mathbf{A}\cdot\mathbf{n}\,dS,\\ &= -\mathbb{F}_N^{(out)} - \textstyle\sum_{k=1}^{N-1}\Big(\mathbb{F}_k + \mathbb{F}_k^{(out)}\Big).\end{aligned} \tag{4.21}$$

Because of the compatibility condition and our assumption on the flux over the inner boundaries we have

$$\sum_{j=1}^{N}\Big(\mathbb{F}_j + \mathbb{F}_j^{(out)}\Big) = 0.$$

Therefore, (4.21) equals to $\mathbb{F}_N = -\Big(\sum_{j=1}^{N}\mathbb{F}_j^{(out)} + \sum_{j=1}^{(N-1)}\mathbb{F}_j\Big)$.

□

Remark 4.7. *Note that,*

(i) if the domain contains only one outlet, i.e. $N = 1$, one cannot prescribe the flux of the outlet due to the compatibility condition $\mathbb{F} = -\mathbb{F}^{(out)}$;

(ii) if the domain contains only a pair symmetric outlets (not the case of two self-symmetric outlets), it holds that $\mathbb{F}_1 = \mathbb{F}_2$. Therefore, one cannot prescribe the flux in either outlet due to $\mathbb{F}_i = -\frac{1}{2}\mathbb{F}^{(out)}$, $i = 1,2$.

Remark 4.8. *If we assume that all the data to be symmetric with respect to the x_1-axis, and all the inner boundaries intersect the x_1-axis we can construct a solenoidal extension of the boundary value with arbitrary large flux. The extension* $\mathbf{A}$ *constructed above can be symmetrized by using the formula, for $\mathbf{A} = (A_1, A_2)$ we define $\tilde{\mathbf{A}} = (\tilde{A}_1, \tilde{A}_2)$ as follows:*

$$\begin{aligned}\tilde{A}_1(x) &= \frac{1}{2}\Big(A_1(x_1,x_2) + A_1(x_1,-x_2)\Big), \quad x\in\Omega,\\ \tilde{A}_2(x) &= \frac{1}{2}\Big(A_2(x_1,x_2) - A_2(x_1,-x_2)\Big), \quad x\in\Omega.\end{aligned} \tag{4.22}$$

We shall see this in the following sections.

5 Symmetric Domains with N Outlets

Now we study the problem

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{a} \quad \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

in general symmetric domain Ω containing finitely many outlets. We distinguish two types of domains. The first type of domains contains at least one infinite outer boundary intersecting the x_1-axis. Whereas the second type of domains contains the x_1-axis, i.e. the outer boundary does not intersect the axis of symmetry.

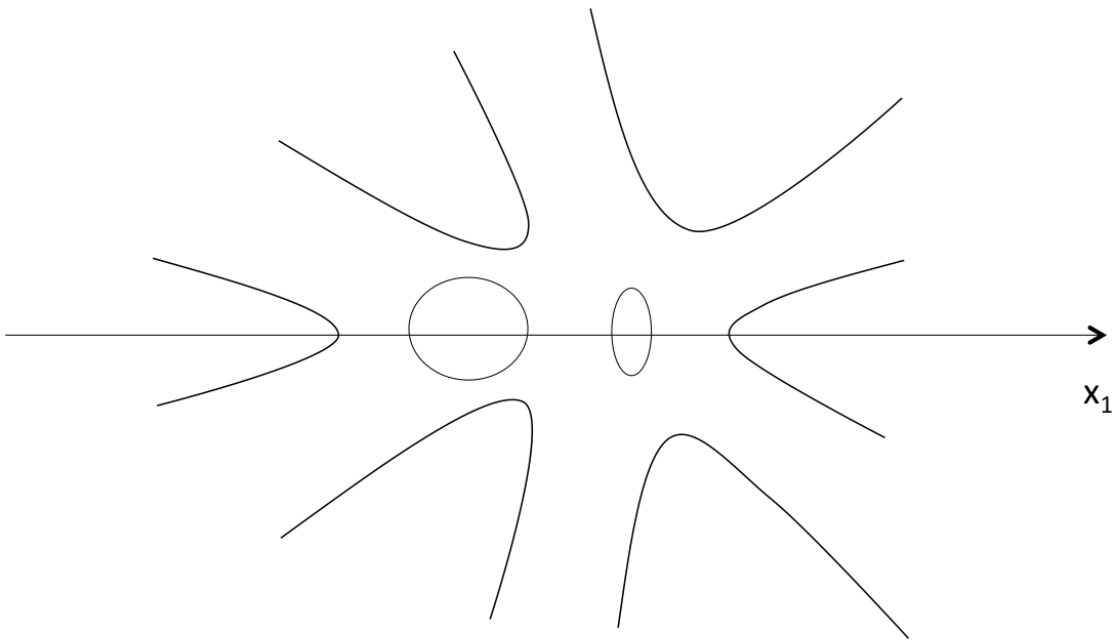


Figure 2.4: General Symmetric Domain Ω

The fluxes over the outer boundaries are

$$\int\limits_{\Gamma_0^m} \mathbf{a}\cdot\mathbf{n}\,dS = \mathbb{F}_m^{(out)}, \quad m = 1,\dots,N, \ ,$$

the fluxes over the inner boundaries are

$$\int\limits_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\,dS = \mathbb{F}_i^{(inn)}, \quad i = 1,...,I,$$

and the fluxes over each cross section of each outlet are

$$\int\limits_{\sigma_j(R)} \mathbf{a}\cdot\mathbf{n}\,dS = \mathbb{F}_j, \quad j = 1,...,N.$$

We only assume the necessary flux condition:

$$\sum_{i=1}^{I} \mathbb{F}_i^{(inn)} + \sum_{m=1}^{N} \mathbb{F}_m^{(out)} + \mathbb{F}_m = 0$$

5.1 The case where the outer boundary intersects the x_1-axis

We start first to construct a symmetric solenoidal extension of the boundary value from the inner boundary which satisfies the Leray-Hopf inequalities. We follow the idea of Fujita [12] for a bounded domain.

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^2$ be an admissible domain with at least one outer boundary Γ_0^1 intersecting the x_1-axis. Let Ω_{k+1}, ω_k be defined as on page 15 and $\mathbf{a} \in \mathbb{W}^{1/2,2}(\Gamma)$ be a symmetric function. Then $\forall \varepsilon > 0$ there exists a symmetric solenoidal extension $\mathbf{A}$ in Ω satisfying the Leray-Hopf inequalities, i.e., for every symmetric solenoidal $\mathbf{w} \in \mathbb{W}^{1,2}_{loc}(\overline{\Omega})$ with $\mathbf{w}|_{\partial\Omega} = 0$ the following estimate*

$$\begin{aligned} &\Big|\int\limits_{\Omega_{k+1}} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{A}dx\Big| \le c\,\varepsilon \int\limits_{\Omega_{k+1}} |\nabla\mathbf{w}|^2\,dx, \\ &\Big|\int\limits_{\omega_k} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{A}dx\Big| \le c\,\varepsilon \int\limits_{\omega_k} |\nabla\mathbf{w}|^2\,dx \end{aligned} \tag{5.2}$$

hold. Moreover,

$$|\mathbf{A}| \le \frac{C(\varepsilon)}{g_j^2(z_1^{(j)})},\ |\nabla\mathbf{A}| \le \frac{C(\varepsilon)}{g_j^3(z_1^{(j)})}, \quad x \in D_j,\ j = 1,\dots,N, \tag{5.3}$$

And it holds

$$\int\limits_{\sigma_j(R)} \mathbf{A}\cdot\mathbf{n}dS = \mathbb{F}_j, \quad j = 1,\cdots,N.$$

Proof. In order not to lose the main idea in technical details, we assume that there is only one connected inner boundary, i.e. Γ_1. In general the same construction works for domains with finitely many inner boundaries (see Remark 5.9 at the end of this section). We construct firstly a solenoidal function in Ω_0 which takes the flux of the boundary value $\mathbf{a}$ from Γ_1 to Γ_0^1. Then we reduce our problem to the boundary value problem with the stringent outflow condition in a bounded domain Ω_0, we apply the Lemma 4.2 to the boundary value having zero flux on Γ_1, to Ω.

Let $0 < \kappa < 1/2$ be a number and $\beta_\kappa(t) \in C_0^\infty(-\infty,+\infty)$ be an even function such that $\beta_\kappa(t) \le \dfrac{1}{t}$, for $0 < t < +\infty$ and

$$\beta_\kappa(t) = \begin{cases} 0, & 1 \le t < +\infty, \\ \dfrac{1}{t}, & \kappa \le t \le 1/2. \end{cases}$$

Define $y(\kappa) = \int\limits_{-\infty}^{\infty} \beta_\kappa(t)dt$. Note that $y(\kappa) \to +\infty$ as $\kappa \to 0$. Let δ be another small positive number. Then we define a smooth positive function $s(\kappa, t) = \frac{1}{y(\kappa)\,\delta}\beta_\kappa\Big(\frac{t}{\delta}\Big)$. Note that $\operatorname{supp} s \subset [-\delta, \delta]$. Moreover, we have

$$\begin{cases} \int\limits_{-\infty}^{\infty} s(\kappa,t)\,dt = \int\limits_{-\delta}^{\delta} s(\kappa,t)\,dt = 1, \\ 0 \le s(t) \le \dfrac{1}{y(\kappa)\delta}\cdot\dfrac{\delta}{|t|} = \dfrac{1}{y(\kappa)|t|}, \quad t \neq 0. \end{cases}$$

Therefore, we have

$$\sup_t |t| s(\kappa, t) \to 0 \quad \text{as } \kappa \to 0. \tag{5.4}$$

Now we construct a thin strip connecting the inner boundary Γ_1 and one outer boundary intersects the x_1-axis, say Γ_0^1 (see Fig. 2.5). We define a solenoidal function compactly supported in the thin strip. This so called virtue drain function will take the flux from Γ_1 to Γ_0^1. Define a thin strip $\Upsilon = [x_0 + \eta, x_1 - \eta] \times [-\delta, \delta]$, where η is a small positive number.

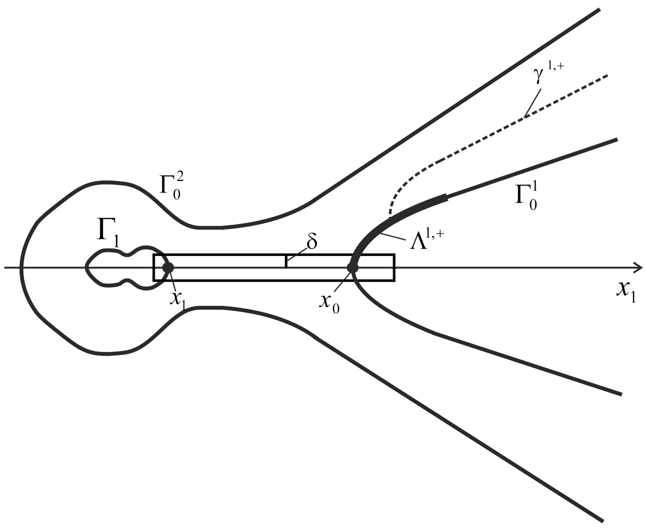


Figure 2.5: Domain Ω

Set

$$\mathbf{b}^{(inn)}(x) = \begin{cases} (-\mathbb{F}_1^{(inn)} s(x_2), 0) & \text{in } \Upsilon, \\ \\ (0,0) & \text{in } \overline{\Omega} \setminus \Upsilon. \end{cases}$$

Since the vector field $\mathbf{b}^{(inn)}$ is solenoidal and vanishes on the upper and lower boundaries of Υ, we have

$$\begin{aligned}
0 &= \int\limits_{\Upsilon\cap G_1} \operatorname{div} \mathbf{b}^{(inn)}\, dx = \int\limits_{\partial(\Upsilon\cap G_1)} \mathbf{b}^{(inn)} \cdot \mathbf{n}\, dS \\
&= \int\limits_{\Upsilon\cap\Gamma_1} \mathbf{b}^{(inn)} \cdot \mathbf{n}\, dS + \int\limits_{(x_1-\eta)\times[-\delta,\delta]} \mathbf{b}^{(inn)} \cdot \mathbf{n}\, dS \\
&= \int\limits_{\Upsilon\cap\Gamma_1} \mathbf{b}^{(inn)} \cdot \mathbf{n}\, dS + \mathbb{F}_1^{(inn)} \int\limits_{-\delta}^{\delta} s(x_2)\, dx_2 \\
&= \int\limits_{\Upsilon\cap\Gamma_1} \mathbf{b}^{(inn)} \cdot \mathbf{n}\, dS + \mathbb{F}_1^{(inn)},
\end{aligned}$$

where G_1 is the simply connected domain enclosed by Γ_1 as introduced in Definition 2.5. Notice further that the outward normal vector on $\partial(\Upsilon \cap G_1)|_{\Gamma_1}$ shows the opposite direction than the one on Γ_1. Therefore, it follows that

$$\int\limits_{\Gamma_1} \mathbf{b}^{(inn)} \cdot \mathbf{n} dS = \mathbb{F}_1^{(inn)}.$$

Let

$$\mathbf{h} = \mathbf{a} - \mathbf{b}^{(inn)}.$$

Then we have

$$\int\limits_{\Gamma_1} \mathbf{h} \cdot \mathbf{n}\, dS = \int\limits_{\Gamma_1} (\mathbf{a} - \mathbf{b}^{(inn)}) \cdot \mathbf{n}\, dS = 0. \tag{5.5}$$

Because of the condition (5.5), Lemma 4.2 and Remark 4.8 there exists a symmetric extension $\mathbf{A}_0$ of the function $\mathbf{h}$ such that $\operatorname{supp} \mathbf{A}_0$ is contained in a small neighborhood of the support of $\mathbf{a}$ and in the strip connecting Γ_1 and Γ_0^1.

$$\operatorname{div} \mathbf{A}_0 = 0, \quad \mathbf{A}_0|_{\Gamma_1} = \mathbf{h},$$

and $\mathbf{A}_0$ satisfies the Leray–Hopf inequalities

$$\begin{aligned}
\Big| \int\limits_{\Omega_{k+1}} (\mathbf{w} \cdot \nabla)\mathbf{w} \cdot \mathbf{A}_0 dx \Big| &\le c\,\varepsilon \int\limits_{\Omega_{k+1}} |\nabla \mathbf{w}|^2\, dx, \\
\Big| \int\limits_{\omega_k} (\mathbf{w} \cdot \nabla)\mathbf{w} \cdot \mathbf{A}_0 dx \Big| &\le c\,\varepsilon \int\limits_{\omega_k} |\nabla \mathbf{w}|^2\, dx.
\end{aligned} \tag{5.6}$$

Set

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{b}^{(inn)},$$

Then $\mathbf{A}$ is a suitable extension of the boundary value $\mathbf{a}$. It remains to prove that $\mathbf{b}^{(inn)}|_{\Upsilon\cap\overline{\Omega}}$ satisfies the Leray-Hopf inequalities. For the sake of simplicity we keep the notation $\mathbf{b}^{(inn)}$ instead of $\mathbf{b}^{(inn)}|_{\Upsilon\cap\overline{\Omega}}$ for the rest of this thesis and we extend it by zero into the whole domain Ω.

Let $\mathbf{w} = (w_1, w_2) \in \mathbb{W}^{1,2}_{loc}(\overline{\Omega})$, $\mathbf{w}|_{\partial\Omega} = 0$, be a symmetric and solenoidal vector field. Then we use the well known following equality

$$(\mathbf{w} \cdot \nabla)\mathbf{w} = \nabla(\frac{1}{2}|\mathbf{w}|^2) + (\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2})(-w_2, w_1). \tag{5.7}$$

Due to (5.7) we have

$$\Big|\int\limits_{\Omega_{k+1}} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{b}^{(inn)}dx\Big| \quad = |\mathbb{F}_1^{(inn)}| \int\limits_{\Upsilon\cap\overline{\Omega}} \Big|\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\Big|\frac{1}{|x_2|}|w_2|\,|x_2|\,s(x_2)dx$$

$$\le |\mathbb{F}_1^{(inn)}|\sup_{x_2}(|x_2|s(x_2)) \int\limits_{\Upsilon\cap\overline{\Omega}} \Big|\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\Big|\frac{|w_2|}{|x_2|}dx.$$

By applying the Hölder inequality, the expression above is less than

$$|\mathbb{F}_1^{(inn)}|\sup_{x_2}(|x_2|s(x_2))\,(\int\limits_{\Upsilon\cap\overline{\Omega}}\Big|\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\Big|^2dx)^{1/2}\cdot(\int\limits_{\Upsilon\cap\overline{\Omega}}\frac{|w_2|^2}{|x_2|^2}dx)^{1/2}.$$

Furthermore,

$$\int\limits_{\Upsilon\cap\overline{\Omega}}\Big|\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\Big|^2dx \le 2\int\limits_{\Omega_{k+1}}|\nabla\mathbf{w}|^2dx\ ,$$

and applying [1] the Hardy inequality we obtain the following estimate

$$\int\limits_{\Upsilon\cap\overline{\Omega}}\frac{|w_2|^2}{|x_2|^2}dx \le \int\limits_{\Omega_{k+1}}|\nabla\mathbf{w}|^2dx.$$

Since $\sup\limits_{0\le t\le\eta}(|t|s(\kappa,t))$ goes to zero as κ goes to zero, we can choose κ small such that $\sup\limits_{x_2}(|x_2|s(x_2))$ is less than $C\varepsilon$. Finally, we obtain the Leray-Hopf inequality[2] (5.2). □

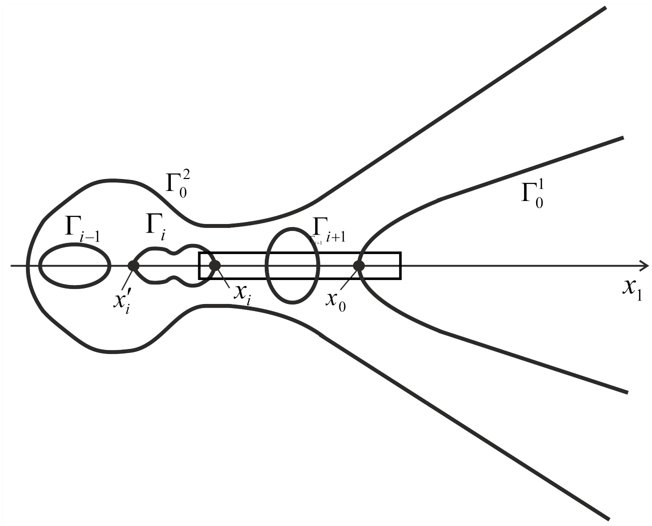


Figure 2.6: Domain Ω

[1]Here we use the fact that the second component w_2 of $\mathbf{w}$ vanishes on the x_1-axis (in trace sense).
[2]Notice that the integral in (5.2) over ω_k is equal to zero since $\mathbf{B}^{(inn)} = 0$ in ω_k.

Remark 5.9. The same construction works for domains with finitely many inner boundaries. For each Γ_i, $i = 1, \ldots, I$, the x_1-axis intersects Γ_i at two points $(x_i, 0)$ and $(x_i', 0)$, where $x_i > x_i'$. In the same manner as before we can construct a thin strip $[x_0 + \eta, x_i - \eta] \times [-\delta, \delta]$ connecting Γ_0^1 and Γ_i, $i = 1, ..., I$ (see Fig. 2.6). We define a function $\mathbf{b}_i^{(inn)}$ on the thin strip as we did for Γ_1, which satisfies

$$\int\limits_{\Gamma_j} \mathbf{b}_i^{(inn)} \cdot \mathbf{n}\, dS = \begin{cases} \mathbb{F}_i^{(inn)}, \ j = i, \\ 0, \ j \neq i, \ j = 1, ..., I. \end{cases}$$

Note that the fluxes of $\mathbf{b}_i^{(inn)}$ across Γ_j cancel each other for $j < i$ and $\mathbf{b}_i^{(inn)}$ vanishes on Γ_j for $j > i$.

Set $\mathbf{b}^{(inn)} = \sum\limits_{i=1}^{I} \mathbf{b}_i^{(inn)}$. Then

$$\int\limits_{\Gamma_i} (\mathbf{a} - \mathbf{b}^{(inn)}) \cdot \mathbf{n}\, dS = 0, \quad i = 1, \ldots, I.$$

Using the usual technique we can find a solenoidal extension $\mathbf{A}_0^{(inn)}$ of the function $\big(\mathbf{a} - \mathbf{b}^{(inn)}\big)|_{\cup_{i=1}^{I}\Gamma_i}$ satisfying the Leray-Hopf inequalities. Then

$$\mathbf{A} = \mathbf{b}^{(inn)} + \mathbf{A}_0$$

is a suitable extension of the boundary value $\mathbf{a}$.

5.2 The case where the outer boundary does not intersect the x_1-axis

In the previous section we studied the problem

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{a} \quad \text{on } \partial\Omega \end{cases} \tag{5.8}$$

in a symmetric domain $\Omega \subset \mathbb{R}^2$, where at least one outer boundary Γ_0^1 intersects the x_1-axis. In that case we can remove all the fluxes from the inner boundaries to Γ_0^1 and apply Lemma 4.2 to obtain a suitable extension. Note that for the symmetric case we do not impose any restrictions on the fluxes over each connected component of the inner boundary. Only the necessary compatibility is needed.

In this section we consider problem (5.8) in a symmetric domain which contains the x_1-axis, i.e. there is no outer boundary intersects the x_1-axis. We denote the semi-infinite outlet containing the positive part of the x_1-axis by $D_1 = \{x \in \Omega : \ |x_2| < g(x_1), x_1 > R_0\}$, for simplicity we omit the index of g_1, and $\Omega = \Omega_0 \bigcup \bigcup_{m=1}^{N} D_m$ where Ω_0 is the bounded part.

Lemma 5.4. *Let Ω be an admissible domain containing the x_1-axis. Assume that the boundary value* $\mathbf{a}$ *is a symmetric function in $\mathbb{W}^{1/2,2}(\partial\Omega)$ having a compact support. Then for every $\varepsilon > 0$ there exists a symmetric solenoidal extension* $\mathbf{A}$ *in Ω satisfying the Leray-Hopf inequalities, i.e., for every symmetric solenoidal function* $\mathbf{w} \in \mathbb{W}^{1,2}_{loc}(\overline{\Omega})$ *with $\mathbf{w}|_{\partial\Omega} = 0$ the following estimates*

$$\begin{aligned} \Big| \int\limits_{\Omega_{k+1}} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{A}\,dx \Big| &\le c\,\varepsilon \int\limits_{\Omega_{k+1}} |\nabla\mathbf{w}|^2\,dx, \\ \Big| \int\limits_{\omega_k} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{A}\,dx \Big| &\le c\,\varepsilon \int\limits_{\omega_k} |\nabla\mathbf{w}|^2\,dx. \end{aligned} \tag{5.9}$$

Moreover,

$$|\mathbf{A}| \le \frac{C(\varepsilon)}{g_j^2(z_1^{(j)})},\ |\nabla\mathbf{A}| \le \frac{C(\varepsilon)}{g_j^3(z_1^{(j)})}, \quad x \in D_j,\ j = 1,\ldots,N, \tag{5.10}$$

and it holds

$$\int\limits_{\sigma_j(R)} \mathbf{A}\cdot\mathbf{n}\,dS = \mathbb{F}_j, \quad j = 1,\cdots,N.$$

The constant c is independent of k and ε.

Remark 5.10. The constant c in (5.9) is of the type

$$c_1 \sum_{i=0}^{N} |\mathbb{F}_i| = c_1 \sum_{i=0}^{N} \Big| \int\limits_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\,dS \Big| \le c_2\,\|\mathbf{a}\|_{W^{1/2,\,2}(\partial\Omega)},$$

where c_2 is independent of $\mathbf{a}$.

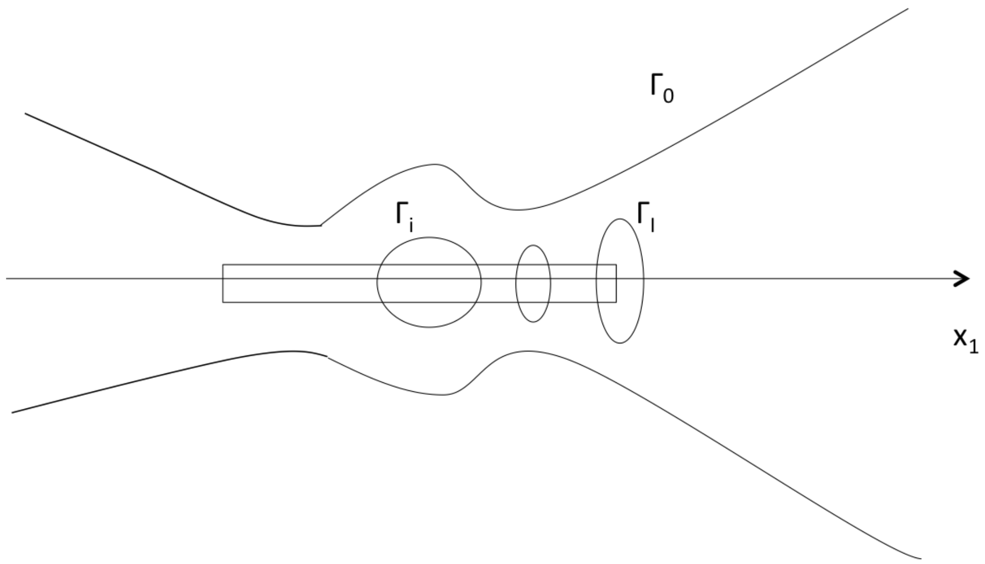


Figure 2.7: Domain Ω with x_1-axis not intersecting outer boundary

We construct a symmetric extension $\mathbf{A}$ of the boundary value $\mathbf{a}$ in two parts. We "remove" the fluxes $\mathbb{F}_i^{(inn)}$, $i = 1, ..., I-1$, to the boundary Γ_I and after this step we get the flux $\sum\limits_{i=1}^{I} \mathbb{F}_i^{(inn)}$ on Γ_I. Then by removing it to infinity we get a modified boundary value satisfying the stringent outflow condition on Γ_i, $i = 1, \ldots, I$. Applying then Lemma 4.2 we can find a suitable boundary value extension. The first part of the construction is inspired by some ideas of Fujita [12].

Construction of b

Before we start with the construction we introduce some auxiliary functions.
For $x \in D_1$, $x_2 > 0$, we set (see [6])

$$\xi(x) = \xi(x_1, x_2) = \Psi\Big(\varepsilon \ln \frac{\gamma(g(x_1) - x_2)}{x_2}\Big), \tag{5.11}$$

where $0 \leq \Psi \leq 1$ is a smooth monotone cut-off function:

$$\Psi(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t \geq 1. \end{cases}$$

Lemma 5.5. *ξ is a smooth function vanishing near $x_2 = g(x_1)$ and equal to 1 in a neighbourhood of $x_2 = 0$. Moreover it holds*

$$\Big|\frac{\partial \xi}{\partial x_i}\Big| \leq \frac{c\,\varepsilon}{x_2}, \quad i = 1, 2, \tag{5.12}$$

$$\Big|\frac{\partial \xi}{\partial x_i}\Big| \leq \frac{C(\varepsilon)}{g(x_1)}, \quad \Big|\frac{\partial^2 \xi}{\partial x_i \partial x_j}\Big| \leq \frac{C(\varepsilon)}{g^2(x_1)} \quad i, j = 1, 2, \tag{5.13}$$

where c is independent of ε and $C(\varepsilon)$ denotes a constant depending on ε.

Proof. First one notices that the support of $\nabla \xi$ is contained in the set where

$$1 \leq \frac{\gamma\big(g(x_1) - x_2\big)}{x_2} \leq e^{1/\varepsilon} \;\Leftrightarrow\; \frac{\big(1+\gamma\big)\,x_2}{\gamma} \leq g(x_1) \leq \frac{\big(e^{1/\varepsilon} + \gamma\big)\,x_2}{\gamma}. \tag{5.14}$$

Then since $\xi(x) = \Psi\big(\varepsilon \ln\big(g(x_1) - x_2\big) - \varepsilon \ln x_2 + \varepsilon \ln \gamma\big)$ one gets

$$\frac{\partial \xi}{\partial x_1} = \Psi' \cdot \frac{\varepsilon\, g'(x_1)}{g(x_1) - x_2}, \quad \frac{\partial \xi}{\partial x_2} = \Psi' \cdot \varepsilon\Big(\frac{-1}{g(x_1) - x_2} - \frac{1}{x_2}\Big), \tag{5.15}$$

where Ψ' is taken at the point $\varepsilon \ln \dfrac{\gamma\,\big(g(x_1) - x_2\big)}{x_2}$. Since we assumed that g' is bounded and Ψ' is bounded as well one derives from (5.14), (5.15)

$$\Big|\frac{\partial \xi}{\partial x_1}\Big| \leq \frac{c\,\varepsilon}{x_2}, \quad \Big|\frac{\partial \xi}{\partial x_2}\Big| \leq c\,\varepsilon\,\Big(\frac{1}{g(x_1) - x_2} + \frac{1}{x_2}\Big) \leq \frac{c\,\varepsilon}{x_2}$$

for some constant c and

$$\Big|\frac{\partial \xi}{\partial x_i}\Big| \leq \frac{c\,\varepsilon}{x_2} \leq \frac{c\,\varepsilon\big(\gamma + e^{1/\varepsilon}\big)}{\gamma\, g(x_1)}, \quad i = 1, 2.$$

Thus, it remains only to prove the last inequality of (5.13). Differentiating (5.15) we get

$$\frac{\partial^2\xi}{\partial x_1^2} = \Psi''\Big(\frac{\varepsilon\, g'(x_1)}{g(x_1)-x_2}\Big)^2 + \Psi'\varepsilon\Big(\frac{g''\big(g(x_1)-x_2\big)-g'^{\,2}}{\big(g(x_1)-x_2\big)^2}\Big).$$

Since we assumed that $g''\,g$ is bounded, by (5.14) we obtain

$$\Big|\frac{\partial^2\xi}{\partial x_1^2}\Big| \le \frac{c\,\varepsilon}{\big(g(x_1)-x_2\big)^2} \le \frac{c\,\varepsilon}{x_2^2} \le \frac{c\,\varepsilon\big(\dfrac{\gamma+e^{1/\varepsilon}}{\gamma}\big)^2}{g^2(x_1)} = \frac{C(\varepsilon)}{g^2(x_1)}.$$

Similarly we have

$$\begin{aligned}
\Big|\frac{\partial^2\xi}{\partial x_1\partial x_2}\Big| &= \Big|\Psi''\varepsilon^2\frac{g'(x_1)}{g(x_1)-x_2}\cdot\Big(-\frac{1}{g(x_1)-x_2}-\frac{1}{x_2}\Big)+\Psi'\varepsilon\frac{g'(x_1)}{\big(g(x_1)-x_2\big)^2}\Big|\\
&\le \frac{c\,\varepsilon}{x_2^2} \le \frac{C(\varepsilon)}{g^2(x_1)},\\
\Big|\frac{\partial^2\xi}{\partial x_2^2}\Big| &= \Big|\Psi''\varepsilon^2\Big(-\frac{1}{g(x_1)-x_2}-\frac{1}{x_2}\Big)^2+\Psi'\varepsilon\Big(-\frac{1}{\big(g(x_1)-x_2\big)^2}+\frac{1}{x_2^2}\Big)\Big|\\
&\le \frac{c\,\varepsilon}{x_2^2} \le \frac{C(\varepsilon)}{g^2(x_1)}.
\end{aligned}$$

This completes the proof of the Lemma. □

We set

$$\tilde{\boldsymbol{\xi}}(x) = (\tilde\xi_1,\tilde\xi_2) = \begin{cases} \Big(-\dfrac{\partial\xi(x_1,x_2)}{\partial x_2},\ \dfrac{\partial\xi(x_1,x_2)}{\partial x_1}\Big), & x_2>0,\\[2ex] \Big(-\dfrac{\partial\xi(x_1,-x_2)}{\partial x_2},\ -\dfrac{\partial\xi(x_1,-x_2)}{\partial x_1}\Big), & x_2<0. \end{cases} \tag{5.16}$$

Then we have

Lemma 5.6. $\tilde{\boldsymbol{\xi}}$ *is a smooth solenoidal symmetric vector field such that for any cross section $\sigma(R)$ of D_1 one has*

$$\int\limits_{\sigma(R)} \tilde{\boldsymbol{\xi}}\cdot\mathbf{e}_1\,dx_2 = 2. \tag{5.17}$$

Proof. Since function $\dfrac{\partial\xi}{\partial x_2}$ is even in x_2 we obtain

$$\int\limits_{\sigma(R)} \tilde{\boldsymbol{\xi}}\cdot\mathbf{e}_1\,dx_2 = 2\int\limits_0^{g(x_1)}\Big(-\frac{\partial\xi}{\partial x_2}\Big)\,dx_2 = -2\,\xi(x_1,g(x_1)) + 2\xi(x_1,0) = 2.$$

□

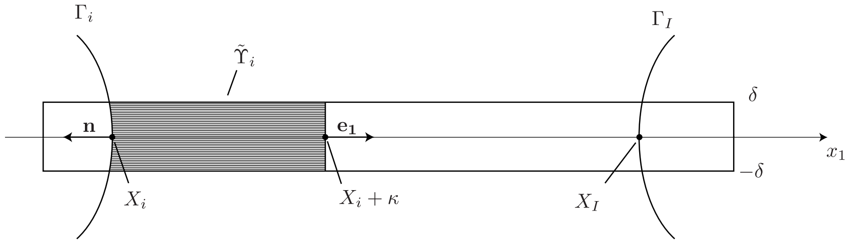


Figure 2.8: The strip Υ_i

Now we start to construct the extension $\mathbf{A}$. Let us choose δ small enough in such a way that the straight line $x_2 = \delta$ cuts each of the Γ_i, $\quad i = 1, ..., I$, at only two points. For $i = 1, ..., I-1$ we define the thin strips $\Upsilon_i = [X_i - \eta_i,\, X_I + \eta_I] \times [-\delta,\, \delta]$, where η_i and η_I are small positive numbers and note that the points $(X_i - \eta_i,\, 0)$ and $(X_I + \eta_I,\, 0)$ are outside of the domain Ω (see Fig. 2.8). Then on each strip $\Upsilon_i \cap \Omega$, $\quad i = 1, ..., I-1$, joining Γ_i to Γ_I we define $\mathbf{b}_i$ in the following way

$$\mathbf{b}_i(x) = -\mathbb{F}_i^{(inn)}\,(\tilde{\xi}_1, \tilde{\xi}_2) = \begin{cases} \dfrac{\mathbb{F}_i^{(inn)}}{2}\Big(\dfrac{\partial \xi_\delta(x_2)}{\partial x_2},\, 0\Big), & \text{in } \Upsilon_i \cap \Omega, \quad x_2 > 0, \\[2ex] \dfrac{\mathbb{F}_i}{2}\Big(\dfrac{\partial \xi_\delta(-x_2)}{\partial x_2},\, 0\Big), & \text{in } \Upsilon_i \cap \Omega, \quad x_2 < 0, \\[2ex] (0,\, 0), \quad \text{in } \overline{\Omega} \setminus (\Upsilon_i \cap \Omega), \end{cases}$$

where

$$\xi_\delta(x) = \Psi\Big(\varepsilon \,\ln \frac{\delta - x_2}{x_2}\Big).$$

Notice that the Lemma 5.5 and Lemma 5.6 are valid if we take $\gamma = 1$ and $g(x_1) = \delta$. Since each vector field $\mathbf{b}_i$ is solenoidal and vanishes on the upper and lower boundaries of Υ_i, we have

$$0 = \int_{\tilde{\Upsilon}_i} \operatorname{div} \mathbf{b}_i \, dx = \int_{\partial \tilde{\Upsilon}_i} \mathbf{b}_i \cdot \mathbf{n}\, dS = \int_{\Gamma_i} \mathbf{b}_i \cdot \mathbf{n}\, dS + \int_{(X_i+\kappa)\times[-\delta,\delta]} \mathbf{b}_i \cdot \mathbf{e}_1\, dS$$

$$= \int_{\Gamma_i} \mathbf{b}_i \cdot \mathbf{n}\, dS - \frac{\mathbb{F}_i}{2} \cdot 2 \int_0^\delta \Big(-\frac{\partial \xi_\delta}{\partial x_2}\Big)\, dx_2 = \int_{\Gamma_i} \mathbf{b}_i \cdot \mathbf{n}\, dS - \mathbb{F}_i, \quad \forall i = 1, ..., I-1,$$

where $\tilde{\Upsilon}_i$ is the domain enclosed by Γ_i, $(X_i + \kappa) \times [-\delta, \delta]$ (κ is a small positive number) and the lines $x_2 = \delta$, $x_2 = -\delta$ (see Fig. 2.8). Therefore, it follows that if $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega$ on Γ_i one has

$$\int_{\Gamma_i} \mathbf{b}_i \cdot \mathbf{n}\, dS = \mathbb{F}_i^{(inn)}, \quad \forall i = 1, ..., I-1.$$

We set

$$\mathbf{b} = \sum_{i=1}^{I-1} \mathbf{b}_i.$$

Clearly $\mathbf{b}$ is a symmetric solenoidal vector field. Moreover for every $i = 1, ..., I-1$ one has (note that the flux of $\mathbf{b}_i$ vanishes on Γ_j for every $i \neq j$)

$$\int_{\Gamma_i} (\mathbf{a} - \mathbf{b}) \cdot \mathbf{n}\, dS = \int_{\Gamma_i} (\mathbf{a} - \mathbf{b}_i) \cdot \mathbf{n}\, dS = \mathbb{F}_i^{(inn)} - \mathbb{F}_i^{(inn)} = 0. \tag{5.18}$$

Notice that

$$\int_{\Gamma_I} (\mathbf{a} - \mathbf{b}) \cdot \mathbf{n}\, dS = \sum_{i=1}^{I} \mathbb{F}_i^{(inn)} = \mathbb{F}^{(inn)}.$$

It remains to prove that $\mathbf{b}$ satisfies the Leray-Hopf inequalities. It is enough to prove that each $\mathbf{b}_i$, $i = 1, \cdots, I-1$, satisfies the Leray-Hopf inequalities.

Let $\mathbf{w} = (w_1, w_2) \in \mathbb{W}^{1,2}_{loc}(\overline{\Omega})$, $\mathbf{w}|_{\partial\Omega} = 0$, be a symmetric and solenoidal vector field. Then

$$\int_{\Omega_{k+1}} (\mathbf{w} \cdot \nabla)\, \mathbf{w} \cdot \mathbf{b}_i\, dx = -\mathbb{F}_i^{(inn)} \int_{\Upsilon_i \cap \overline{\Omega}} \big(w_1 \frac{\partial w_1}{\partial x_1} + w_2 \frac{\partial w_1}{\partial x_2}\big)\, \tilde{\xi}_1(x_2)\, dx$$

Since $\dfrac{w_1^2(x)}{2}\, \tilde{\xi}_1(x_2)$ vanishes on the boundary of $\Upsilon_i \cap \overline{\Omega}$ one has

$$\int_{\Upsilon_i \cap \overline{\Omega}} w_1(x) \frac{\partial w_1(x)}{\partial x_1}\, \tilde{\xi}_1(x_2)\, dx = \frac{1}{2} \int_{\Upsilon_i \cap \overline{\Omega}} \frac{\partial \big(w_1^2(x)\, \tilde{\xi}_1(x_2)\big)}{\partial x_1}\, dx = 0.$$

Therefore, using the definition of $\tilde{\xi}_1$, applying the estimate (5.12) and the Hardy[3] type inequality one gets

$$\Big| \int_{\Omega_{k+1}} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{b}_i\, dx \Big| \leq |\mathbb{F}_i^{(inn)}| \int_{\Upsilon_i \cap \Omega} \Big| w_2 \frac{\partial w_1}{\partial x_2}\, \tilde{\xi}_1 \Big|\, dx$$

$$\leq c\,\varepsilon |\mathbb{F}_i^{(inn)}| \int_{\Upsilon_i \cap \Omega} \frac{|w_2|}{|x_2|} \Big| \frac{\partial w_1}{\partial x_2} \Big|\, dx \leq c\,\varepsilon |\mathbb{F}_i^{(inn)}| \int_{\Omega_{k+1}} |\nabla \mathbf{w}|^2\, dx.$$

Construction of the Extension $\mathbf{b}_\infty$.

After moving all the fluxes through Γ_i, $i = I, ..., I-1$, to the last inner boundary Γ_I we need to drain the flux from Γ_I to infinity. There we consider a function g as in Lemma 5.5 such that the outlets D_1 has the form $D_1 = \{x \in \mathbb{R}^2 : |x_2| < g(x_1),\ x_1 > R_0\}$. Note that we treated domains with more general boundaries in [6]. Suppose further that γ is chosen such that

$$\text{the curve } \ x_2 = \frac{\gamma}{\gamma+1}\, g(x_1) \ \text{ crosses } \ \Gamma_I. \tag{5.19}$$

Let us introduce the vector field

[3] For the application of the Hardy type inequality we used the fact that w_2 vanishes on $x_2 = 0$.

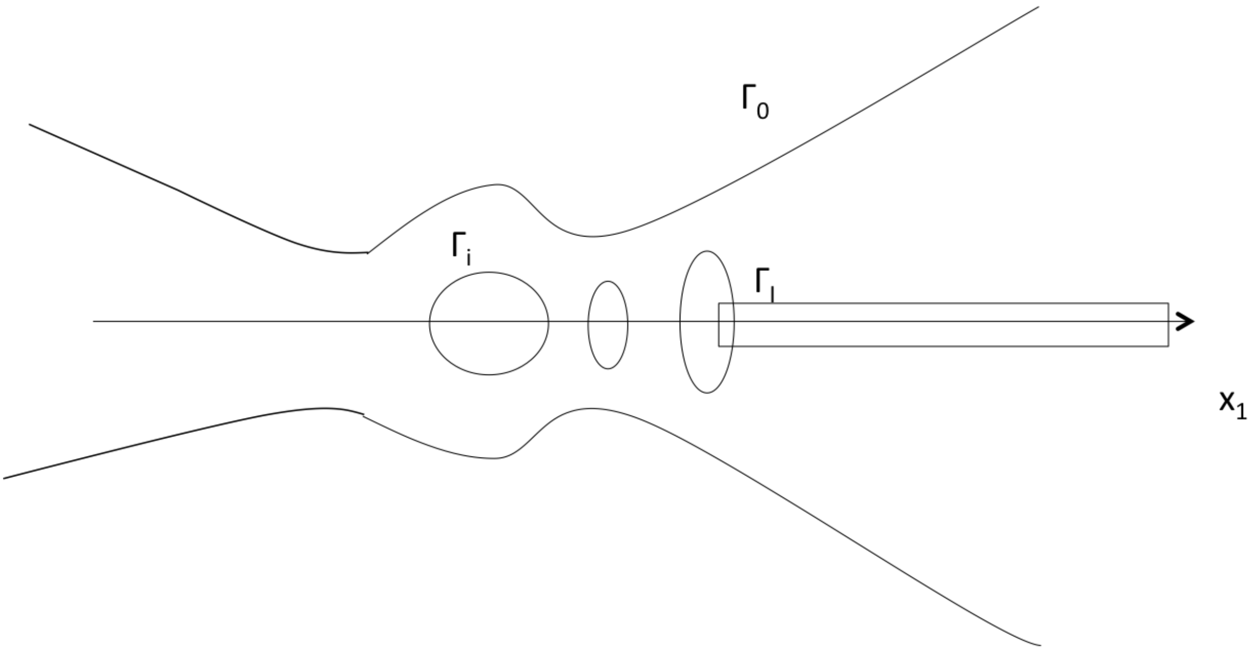


Figure 2.9: Construction of $\mathbf{b}_\infty$

$$\mathbf{b}_\infty(x) = -\mathbb{F}^{(inn)}\,\tilde{\boldsymbol{\xi}} = \begin{cases} \dfrac{\mathbb{F}^{(inn)}}{2}\Big(\dfrac{\partial\xi(x_1,x_2)}{\partial x_2},\ -\dfrac{\partial\xi(x_1,x_2)}{\partial x_1}\Big), & x_2>0,\\[2ex] \dfrac{\mathbb{F}^{(inn)}}{2}\Big(\dfrac{\partial\xi(x_1,-x_2)}{\partial x_2},\ \dfrac{\partial\xi(x_1,-x_2)}{\partial x_1}\Big), & x_2<0,\end{cases}$$

where ξ is defined by (5.11) for x in the thin strip and extended by 0 into D_1. Then since for any cross section σ

$$\int\limits_{\Gamma_I}\mathbf{b}_\infty\cdot\mathbf{n}\,dS = -\int\limits_{\sigma}\mathbf{b}_\infty\cdot\mathbf{n}\,dS = -\frac{\mathbb{F}^{(inn)}}{2}\cdot 2\int\limits_0^{\frac{\gamma}{\gamma+1}g(x_1)}\frac{\partial\xi}{\partial x_2}\,dx_2 =$$

$$= -\mathbb{F}^{(inn)}\Big(\xi(x_1,\,\tfrac{\gamma}{\gamma+1}\,g(x_1)) - \xi(x_1,\,0)\Big) = \mathbb{F}^{(inn)},$$

one has

$$\int\limits_{\Gamma_I}\big(\mathbf{a}-\mathbf{b}-\mathbf{b}_\infty\big)\,\mathbf{n}\,dS = 0. \tag{5.20}$$

Since the support of $\mathbf{b}$ does not intersect the inner boundaries we also have

$$\int\limits_{\Gamma_i}\big(\mathbf{a}-\mathbf{b}-\mathbf{b}_\infty\big)\,\mathbf{n}\,dS = 0, \quad i=1,\dots,I-1. \tag{5.21}$$

Because of (5.20), (5.21), Lemma 4.2 and Remark 4.8 there exists a symmetric extension $\mathbf{A}_0$ of $\big(\mathbf{a}-\mathbf{b}-\mathbf{b}_\infty\big)\Big|_{\partial\Omega}$ such that $\operatorname{supp}\mathbf{A}_0$ is contained in a small neighborhood of Γ_i, $i=1,\dots,I$.

$$\operatorname{div}\mathbf{A}_0 = 0, \quad \mathbf{A}_0|_{\Gamma_I} = \big(\mathbf{a}-\mathbf{b}-\mathbf{b}_\infty\big)\Big|_{\Gamma_I},$$
$$\mathbf{A}_0|_{\Gamma_i} = \big(\mathbf{a}-\mathbf{b}\big)\Big|_{\Gamma_i}, i=1,\dots,I-1.$$

and $\mathbf{A}_0$ satisfies the Leray–Hopf inequalities for every solenoidal function $\mathbf{w} \in \mathbb{W}^{1,2}_{loc}(\overline{\Omega})$ with $\mathbf{w}|_{\partial\Omega} = 0$

$$
\begin{aligned}
&\Big|\int\limits_{\Omega_{k+1}} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{A}_0\,dx\Big| \le c\,\varepsilon \int\limits_{\Omega_{k+1}} |\nabla\mathbf{w}|^2\,dx,\\
&\Big|\int\limits_{\omega_k} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{A}_0\,dx\Big| \le c\,\varepsilon \int\limits_{\omega_k} |\nabla\mathbf{w}|^2\,dx
\end{aligned}
\tag{5.22}
$$

with a constant c independent of k and ε. Then

$$\mathbf{A} = \mathbf{b} + \mathbf{b}_\infty + \mathbf{A}_0$$

is a symmetric solenoidal extension of $\mathbf{a}$. It remains to prove that $\mathbf{A}$ satisfies the Leray-Hopf inequalities. It is enough to prove that $\mathbf{b}_\infty$ satisfies the Leray-Hopf inequalities.

Let $\mathbf{w} = (w_1, w_2) \in \mathbb{W}^{1,2}_{loc}(\overline{\Omega})$, $\mathbf{w}|_{\partial\Omega} = 0$, be a symmetric and solenoidal vector field. We use the well known identity

$$(\mathbf{w}\cdot\nabla)\,\mathbf{w} = \nabla\big(\frac{1}{2}|\mathbf{w}|^2\big) + \big(\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}\big)\,(-w_2,\, w_1). \tag{5.23}$$

Since $\mathbf{b}_\infty$ is solenoidal, it is L^2- orthogonal to the first term of the right-hand side of (5.23). Then one obtains

$$
\begin{aligned}
&\Big|\int\limits_{\Omega_{k+1}} (\mathbf{w}\cdot\nabla)\,\mathbf{w}\,\cdot\mathbf{b}_\infty\,dx\Big|\\
&\le |\mathbb{F}| \int\limits_{\Omega_{k+1}} \Big|\big(\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}\big)\,(-w_2\,\tilde{\xi}_1 + w_1\,\tilde{\xi}_2)\Big|\,dx\\
&\le |\mathbb{F}| \Big(\int\limits_{\Omega_{k+1}} |\nabla\mathbf{w}|^2 dx\Big)^{1/2} \Big(\int\limits_{\Omega_{k+1}} |(-w_2\,\tilde{\xi}_1 + w_1\,\tilde{\xi}_2)|^2 dx\Big)^{1/2}.
\end{aligned}
\tag{5.24}
$$

Let $G^{\pm}$ denotes the curve $x_2 = \pm g(x_1)$. Then using (5.12), (5.15) for $x \in \Omega$ and $x_2 > 0$, we have

$$|\tilde{\boldsymbol{\xi}}_1| = \Big|\frac{\partial \xi}{\partial x_2}\Big| \le c\varepsilon \frac{1}{x_2}, \qquad |\tilde{\boldsymbol{\xi}}_2| = \Big|\frac{\partial \xi}{\partial x_1}\Big| \le \frac{c\,\varepsilon}{\operatorname{dist}(x, G^+)}. \tag{5.25}$$

Therefore, from (5.24) and (5.25) applying[4] Hardy type inequality (see Lemma 1.5) we get

$$
\begin{aligned}
&\Big|\int\limits_{\Omega^+_{k+1}} (\mathbf{w}\cdot\nabla)\,\mathbf{w}\,\cdot\mathbf{b}_\infty\,dx\Big|\\
&\le c\,\varepsilon\,|\mathbb{F}|\Big(\int\limits_{\Omega^+_{k+1}} |\nabla\mathbf{w}|^2 dx\Big)^{1/2}\Big[\Big(\int\limits_{\Omega^+_{k+1}} \frac{|w_2|^2}{|x_2|^2}\,dx\Big)^{1/2} + \Big(\int\limits_{\Omega^+_{k+1}} \frac{|w_1|^2}{\operatorname{dist}^2(x, G^+)}\,dx\Big)^{1/2}\Big]\\
&\le c\,\varepsilon\,|\mathbb{F}| \int\limits_{\Omega^+_{k+1}} |\nabla\mathbf{w}|^2\,dx,
\end{aligned}
$$

[4]Here we used the fact that $w_2 = 0$ on $x_2 = 0$ and we supposed that $\mathbf{w}$ is extended by 0 outside Ω.

where $\Omega_{k+1}^{+} = \{x \in \Omega_{k+1} :\ x_2 > 0\}$. The same estimate is valid in Ω_{k+1}^{-}. Therefore, $\mathbf{b}_\infty$ satisfies the Leray-Hopf inequalities for every symmetric solenoidal function $\mathbf{w} \in \mathbb{W}^{1,2}_{loc}(\overline{\Omega})$ with $\mathbf{w}|_{\partial\Omega} = 0$

$$\begin{aligned}
&\Big|\int\limits_{\Omega_{k+1}} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{b}_\infty\,dx\Big| \le c\,\varepsilon \int\limits_{\Omega_{k+1}} |\nabla\mathbf{w}|^2\,dx,\\
&\Big|\int\limits_{\omega_k} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{b}_\infty\,dx\Big| \le c\,\varepsilon \int\limits_{\omega_k} |\nabla\mathbf{w}|^2\,dx.
\end{aligned} \tag{5.26}$$

Moreover, one has the estimates

$$|\mathbf{b}_\infty| \le \frac{C(\varepsilon)}{g(x_1)}, \qquad |\nabla\mathbf{b}_\infty| \le \frac{C(\varepsilon)}{g^2(x_1)}, \quad x \in D_1. \tag{5.27}$$

We proved Lemma 5.4.

6 Existence Theorem

We prove in this section our main result Theorem 2.1. In the previous sections we construct solenoidal extensions of boundary values in different different types of domains, which satisfy also the Leray-Hopf inequalities (2.9). Therefore, we now look for a symmetric solution $\mathbf{u}$ in the form

$$\mathbf{u}(x) = \mathbf{A}(x,\varepsilon) + \mathbf{v}(x), \tag{6.1}$$

where $\mathbf{A}$ is an extension of the boundary value $\mathbf{a}$ constructed in the previous sections (see Lemma 5.4). For the sake of simplicity we assume the domain contains only one outlet, we therefore can omit the index of the g_j's. The proof for general domain containing N outlet works analogously. In order to prove the existence of at least one weak solution we need some classical results. Let us define $\Omega_{k+1}\setminus\overline{\Omega_k}$ by ω_k.

Lemma 6.1. *(Poincaré inequality). Let $u \in \mathbb{W}^{1,2}_{loc}(\overline{\Omega})$, $u\big|_{\partial\Omega} = 0$. Then the following inequality*

$$\int\limits_{\omega_k} |u(x)|^2\,dx \le c\,g^2(R_k) \int\limits_{\omega_k} |\nabla u(x)|^2\,dx, \tag{6.2}$$

holds, where the constant c is independent of u and k.

For the proof of this lemma recall (2.1):

$$\frac{1}{2}g(R_k) \le g(t) \le \frac{3}{2}g(R_k), \quad t \in [R_k,\ R_{k+1}].$$

Lemma 6.2. *Let $u \in \mathbb{W}^{1,2}_{loc}(\overline{\Omega})$, $u\big|_{\partial\Omega} = 0$. Then the following inequality*

$$\|u\|_{L^4(\omega_k)} \le c\,g^{1/2}(R_k)\|\nabla u\|_{L^2(\omega_k)}, \tag{6.3}$$

holds, where the constant c is independent of u and k.

Proof. The proof of this lemma follows directly from the following inequality

$$\|u\|_{L^4(\omega_k)} \le c\|\nabla u\|^{1/2}_{L^2(\omega_k)} \cdot \|u\|^{1/2}_{L^2(\omega_k)}, \tag{6.4}$$

the estimates (2.1) and the Poincaré inequality (6.2). The constant c in (6.4) is independent of k. □

Lemma 6.3. *Let* $\omega_k = \{x : R_k < x_1 < R_{k+1},\ |x_2| < g(x_1)\}$. *Let* $f \in \mathbb{L}^2(\omega_k)$ *and*

$$\int\limits_{\omega_k} f\,dx = 0.$$

Then the problem

$$\left\{ \begin{array}{rcl} \operatorname{div} \boldsymbol{u} &=& f \ \text{ in } \omega_k, \\ \boldsymbol{u} &=& 0 \ \text{ on } \partial\omega_k \end{array} \right. \tag{6.5}$$

admits a solution $\boldsymbol{u} \in \mathbb{W}^{1,2}_0(\omega_k)$ *satisfying the estimate*

$$\|\nabla \boldsymbol{u}\|_{L^2(\omega_k)} \le c\,\|f\|_{L^2(\omega_k)} \tag{6.6}$$

with the constant c *independent of* $\boldsymbol{u}$, f *and* k.

Remark 6.1. In [57] the family of the domains ω_k was chosen in a special way in order to have solutions of the problem (6.5) satisfying the estimates (6.6) with a constant c independent of k. Below we give a detailed proof of that fact.

Proof. Recall that $R_{k+1} - R_k = \dfrac{g(R_k)}{2L}$ and L is the Lipschitz constant of g. Consider the transformation F defined by

$$y = (y_1, y_2) = F(x) = \left(\frac{2L\,(x_1 - R_k)}{g(R_k)},\ \frac{2L\,x_2}{g(R_k)} \right).$$

Through this transformation ω_k is transformed into a domain $F(\omega_k)$ such that

$$0 \le y_1 = \frac{2L\,(x_1 - R_k)}{g(R_k)} \le \frac{2L\,(R_{k+1} - R_k)}{g(R_k)} = 1,$$

$$|y_2| \le \frac{2L\,g(x_1)}{g(R_k)} = \frac{2L\,(g(x_1) - g(R_k) + g(R_k))}{g(R_k)} \le 2L\left(\frac{L\,(R_{k+1} - R_k)}{g(R_k)} + 1 \right) = 3L.$$

Moreover, the upper and the lower boundary of $F(\omega_k)$ is given by $\pm$ the graph of the function h_k defined as

$$h_k(y_1) = \frac{2L}{g(R_k)}\, g\Big(\frac{g(R_k)}{2L}\, y_1 + R_k \Big), \quad y_1 \in (0,1).$$

Note that h_k satisfies

$$|h_k(y_1) - h_k(y_1')| \le L|y_1 - y_1'| \quad \forall y_1, y_1' \in (0,1).$$

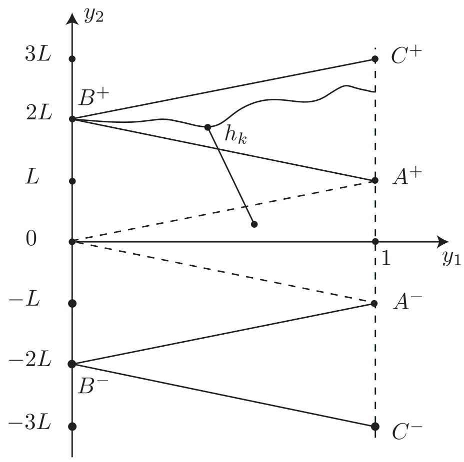


Figure 2.10: Transformed domain

Since $h_k(0) = 2L$ it is clear that the graph of h_k (resp. $-h_k$) is contained in the triangle $A^+B^+C^+$ (resp. $A^-B^-C^-$) (see Fig. 2.5). Any straight line joining a point of the triangle A^-OA^+ (notice that $O = (0,0)$) to the graph of $\pm h_k$ will necessarily have a slope larger than L and thus $F(\omega_k)$ is a star shaped domain with respect to any point of A^-OA^+ and bounded independently of k. One has if $J_F(x)$ denotes the Jacobian determinant of F and F^{-1} the inverse of F

$$\int_{F(\omega_k)} f(F^{-1}(y))\,dy = \int_{\omega_k} f(x)\,|J_F(x)|\,dx = \left(\frac{2L}{g(R_k)}\right)^2 \int_{\omega_k} f(x)\,dx = 0.$$

Thus there exists $\mathbf{v}$ solution to

$$\begin{cases} \operatorname{div} \mathbf{v}(y) &= \dfrac{g(R_k)}{2L} f(F^{-1}(y)) \ \text{ in } F(\omega_k), \\ \mathbf{v}(y) &= 0 \ \text{ on } \partial F(\omega_k) \end{cases}$$

which satisfies (see [35])

$$\|\nabla \mathbf{v}\|_{L^2(F(\omega_k))} \leq c\,\Big\|\frac{g(R_k)}{2L} f(F^{-1}(y))\Big\|_{L^2(F(\omega_k))}, \tag{6.7}$$

where c is independent of k. Set

$$\mathbf{u}(x) = \mathbf{v}(F(x)).$$

One has the summation convention

$$\frac{\partial u_k(x)}{\partial x_i} = \sum_{l=1}^{2} \frac{\partial v_k(F(x))}{\partial y_l} \cdot \frac{\partial y_l}{\partial x_i} = \frac{2L}{g(R_k)} \cdot \frac{\partial v_k(F(x))}{\partial y_i}.$$

Thus $\mathbf{u}$ satisfies

$$\begin{cases} \operatorname{div} \mathbf{u}(x) &= \dfrac{2L}{g(R_k)} \operatorname{div} \mathbf{v}(F(x)) = f(x) \ \text{ in } \omega_k, \\ \mathbf{u}(x) &= 0 \ \text{ on } \partial\omega_k. \end{cases}$$

Moreover,

$$\|\nabla \mathbf{u}\|_{L^2(\omega_k)} = \frac{2L}{g(R_k)} \|\nabla \mathbf{v}(F(x))\|_{L^2(\omega_k)}.$$

Since (see (6.7))

$$\|\nabla \mathbf{v}(F(x))\|^2_{L^2(\omega_k)} = \int_{\omega_k} |\nabla \mathbf{v}(F(x))|^2 dx = \int_{F(\omega_k)} |\nabla \mathbf{v}(y)|^2 \Big(\frac{g(R_k)}{2L}\Big)^2 dy$$

$$\leq c\,\Big(\frac{g(R_k)}{2L}\Big)^4 \int_{F(\omega_k)} f^2(F^{-1}(y))dy = c\,\Big(\frac{g(R_k)}{2L}\Big)^2 \int_{\omega_k} f^2(x)\,dx$$

the result follows. □

Proof of the Theorem 2.1: we construct a solution to (2.5) as limit of a sequence $\mathbf{v}^{(l)} \in \mathbb{H}_S(\Omega_l)$, where $\mathbf{v}^{(l)}$ are solutions to

$$\begin{aligned} &\nu \int_{\Omega_l} \nabla \mathbf{v}^{(l)} : \nabla \boldsymbol{\eta}\, dx - \int_{\Omega_l} ((\mathbf{A} + \mathbf{v}^{(l)}) \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{v}^{(l)}\, dx - \int_{\Omega_l} (\mathbf{v}^{(l)} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A}\, dx \\ &= \int_{\Omega_l} (\mathbf{A} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A}\, dx - \nu \int_{\Omega_l} \nabla \mathbf{A} : \nabla \boldsymbol{\eta}\, dx + \int_{\Omega_l} \mathbf{f} \cdot \boldsymbol{\eta}\, dx \end{aligned} \tag{6.8}$$

for any test function $\boldsymbol{\eta} \in H_S(\Omega_l)$. Due, for instance, to the Riesz representation theorem there exits a unique element $\mathcal{A}\,\mathbf{v}^{(l)} \in H_S(\Omega_l)$ such that

$$\begin{aligned} &\int_{\Omega_l} \nabla \widehat{\mathcal{A}}\, \mathbf{v}^{(l)} : \nabla \boldsymbol{\eta}\, dx \\ &= \nu^{-1} \Bigg(\int_{\Omega_l} (\mathbf{v}^{(l)} \cdot \nabla)\, \boldsymbol{\eta} \cdot \mathbf{v}^{(l)}\, dx + \int_{\Omega_l} (\mathbf{A} \cdot \nabla)\, \boldsymbol{\eta} \cdot \mathbf{v}^{(l)}\, dx \\ &+ \int_{\Omega_l} (\mathbf{v}^{(l)} \cdot \nabla)\, \boldsymbol{\eta} \cdot \mathbf{A}\, dx + \int_{\Omega_l} (\mathbf{A} \cdot \nabla)\, \boldsymbol{\eta} \cdot \mathbf{A}\, dx + \int_{\Omega_l} \mathbf{f} \cdot \boldsymbol{\eta}\, dx \Bigg) - \int_{\Omega_l} \nabla \mathbf{A} : \nabla \boldsymbol{\eta}\, dx \\ &\forall \boldsymbol{\eta} \in \mathbb{H}_S(\Omega_l). \end{aligned}$$

The equation (6.8) is equivalent to the operator equation

$$\mathbf{v}^{(l)} = \widehat{\mathcal{A}}\, \mathbf{v}^{(l)}. \tag{6.9}$$

It can be proved (see [33]) that the operator $\widehat{\mathcal{A}} : \mathbb{H}_S(\Omega_l) \hookrightarrow \mathbb{H}_S(\Omega_l)$ is compact and the solvability of the operator equations (6.9) can be obtained by applying the Leray–Schauder Theorem. To do this we need to show that the norms of all possible solutions of the operator equations

$$\mathbf{v}^{(l,\lambda)} = \lambda \widehat{\mathcal{A}}\, \mathbf{v}^{(l,\lambda)}, \quad \lambda \in [0,\, 1], \tag{6.10}$$

are bounded by a constant independent of λ. Take $\boldsymbol{\eta} = \mathbf{v}^{(l,\lambda)}$ in (6.10). This yields

$$\begin{aligned} \nu \int_{\Omega_l} |\nabla \mathbf{v}^{(l,\lambda)}|^2\, dx \;&=\; \lambda \int_{\Omega_l} (\mathbf{A} \cdot \nabla)\, \mathbf{v}^{(l,\lambda)} \cdot \mathbf{A}\, dx - \lambda\, \nu \int_{\Omega_l} \nabla \mathbf{A} : \nabla \mathbf{v}^{(l,\lambda)}\, dx \\ &+\; \lambda \int_{\Omega_l} \mathbf{f} \cdot \mathbf{v}^{(l,\lambda)}\, dx + \lambda \int_{\Omega_l} (\mathbf{v}^{(l,\lambda)} \cdot \nabla)\, \mathbf{v}^{(l,\lambda)} \cdot \mathbf{A}\, dx. \end{aligned} \tag{6.11}$$

We estimate the first three terms of the right-hand side of (6.11) by using the Hölder and the Cauchy inequalities, and to estimate the last term of (6.11) we use the Leray–Hopf inequality (5.9). We obtain

$$
\begin{aligned}
&\nu \int_{\Omega_l} |\nabla \mathbf{v}^{(l,\lambda)}|^2 dx \le c\,\mu \int_{\Omega_l} |\nabla \mathbf{v}^{(l,\lambda)}|^2\, dx \\
&+\frac{c}{\mu}\Big(\int_{\Omega_l} |\nabla \mathbf{A}|^2 dx + \int_{\Omega_l} |\mathbf{A}|^4 dx + \|\mathbf{f}\|^2_{H^*(\Omega_l)}\Big) + c(\mathbb{F}_1, ..., \mathbb{F}_N)\,\varepsilon \int_{\Omega_l} |\nabla \mathbf{v}^{(l,\lambda)}|^2\, dx.
\end{aligned}
\tag{6.12}
$$

Taking $\mu = \dfrac{\nu}{4\,c}$ and $\varepsilon = \dfrac{\nu}{4\,c(\mathbb{F}_1, ..., \mathbb{F}_N)}$, we obtain

$$
\frac{\nu}{2}\|\nabla \mathbf{v}^{(l,\lambda)}\|^2_{L^2(\Omega_l)} \le c\Big(\|\nabla \mathbf{A}\|^2_{L^2(\Omega_l)} + \|\mathbf{A}\|^4_{L^4(\Omega_l)} + \|\mathbf{f}\|^2_{H^*(\Omega_l)}\Big).
$$

Since ε is now fixed, we have also (note that $\operatorname{supp} \mathbf{B}_0 \subset \Omega_0$)

$$
\begin{aligned}
&\|\nabla \mathbf{A}\|^2_{L^2(\Omega_l)} = \|\nabla \mathbf{B}_0 + \nabla \mathbf{B}_\infty\|^2_{L^2(\Omega_l)} = \|\nabla \mathbf{B}_0\|^2_{L^2(\Omega_0)} + \|\nabla \mathbf{B}_\infty\|^2_{L^2(\Omega_l)} \\
&\le c\,\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}\Big(\int_{\Omega_0} dx + \int_{\Omega_l} \frac{dx}{g^4(x_1)}\Big) \\
&\le c\,\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}\Big(\operatorname{meas}(\Omega_0) + \int_{R_0}^{R_l}\int_{-g(x_1)}^{g(x_1)} \frac{dx_1\,dx_2}{g^4(x_1)}\Big) \\
&\le c\,\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}\Big(1 + \int_{R_0}^{R_l} \frac{dx_1}{g^3(x_1)}\Big).
\end{aligned}
\tag{6.13}
$$

Similarly (see (5.27))

$$
\|\mathbf{A}\|^4_{L^4(\Omega_l)} \le c\,\|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)}\Big(1 + \int_{R_0}^{R_l} \frac{dx_1}{g^3(x_1)}\Big). \tag{6.14}
$$

The constant c in (6.13) and (6.14) is independent of l.
Therefore, we obtain for all $0 \le \lambda \le 1$

$$
\|\nabla \mathbf{v}^{(l,\lambda)}\|^2_{L^2(\Omega_l)} \le c(\mathbf{a}, \|\mathbf{f}\|_*)\Big(1 + \int_{R_0}^{R_l} \frac{dx_1}{g^3(x_1)}\Big).
$$

Hence, according to the Leray–Schauder Theorem each operator equation (6.9) has at least one weak symmetric solution $\mathbf{v}^{(l)} \in \mathbb{H}_S(\Omega_l)$. These solutions satisfy the integral identity (6.8) and the inequality

$$
\|\nabla \mathbf{v}^{(l)}\|^2_{L^2(\Omega_l)} \le c(\mathbf{a}, \|\mathbf{f}\|_*)\Big(1 + \int_{R_0}^{R_l} \frac{dx_1}{g^3(x_1)}\Big). \tag{6.15}
$$

If $\int\limits_{R_0}^{+\infty} \frac{dx_1}{g^3(x_1)} < +\infty$, the right hand side of the above inequality is bounded by a constant uniformly independent of l. Extending the solutions $\mathbf{v}^{(l)}$ by 0 into $\Omega \setminus \Omega_l$ we get functions $\tilde{\mathbf{v}}^{(l)} \in H_S(\Omega)$. The sequence $\{\tilde{\mathbf{v}}^{(l)}\}$ is bounded in the space $\mathbb{H}_S(\Omega)$. Therefore, there exists a subsequence $\{\tilde{\mathbf{v}}^{(l_m)}\}$ which converges weakly in $\mathbb{H}_S(\Omega)$ and strongly[5] in $\mathbb{L}^4(\Omega_l)$ for any l. Taking in integral identity (6.8) an arbitrary test function $\boldsymbol{\eta}$ with a compact support, we can find a number l such that $\operatorname{supp} \boldsymbol{\eta} \subset \Omega_l$ and $\boldsymbol{\eta} \in \mathbb{H}_S(\Omega_l)$. We can pass in (6.8) to a limit as $l_m \to +\infty$. As a result we get for the limit vector function $\mathbf{v}$ the integral identity (2.5). Obviously, following estimate

$$\int\limits_{\Omega} |\nabla \mathbf{v}|^2 \, dx \le c(\mathbf{a}, \|\mathbf{f}\|_*) \left(1 + \int\limits_{R_0}^{+\infty} \frac{dx_1}{g^3(x_1)}\right)$$

holds.

However, if $\int\limits_{R_0}^{+\infty} \frac{dx_1}{g^3(x_1)} = +\infty$, we cannot pass to a limit because the right hand side of (6.15) is growing. Therefore, we have to control the Dirichlet integral of the vector field $\mathbf{v}^{(l)}$ over subdomains $\Omega_k \subset \Omega_l$, for $k \le l$. To do this we apply the special techniques (so called estimates of Saint Venant type) developed by V.A. Solonnikov and O.A. Ladyzhenskaya (see [36], [58]). Let us estimate the norm $\|\nabla \mathbf{v}^{(l)}\|_{L^2(\Omega_k)}$ with $k < l$. We introduce the function

$$\mathbf{U}_k^{(l)}(x) = \begin{cases} \mathbf{v}^{(l)}(x), & x \in \Omega_k, \\ \theta_k(x)\mathbf{v}^{(l)}(x) + \widehat{\mathbf{v}}_k^{(l)}(x), & x \in \omega_k, \\ 0, & x \in \Omega \setminus \Omega_{k+1}, \end{cases} \tag{6.16}$$

where $\theta_k(x)$ is a smooth even in x_2 cut-off function with the following properties:

$$\theta_k(x) = \begin{cases} 1, & x \in \Omega_k, \\ 0, & x \in \Omega \setminus \Omega_{k+1}, \end{cases}$$

$$|\nabla \theta_k(x)| \le \frac{c}{g(R_k)}. \tag{6.17}$$

Let $\widehat{\mathbf{v}}_k^{(l)}$ be a solution of the problem

$$\begin{aligned} \operatorname{div} \widehat{\mathbf{v}}_k^{(l)} &= -\nabla \theta_k \cdot \mathbf{v}^{(l)} \quad \text{in} \quad \omega_k, \\ \widehat{\mathbf{v}}_k^{(l)} &= 0 \quad \text{on} \quad \partial\omega_k. \end{aligned} \tag{6.18}$$

Since

$$\int\limits_{\omega_k} \nabla \theta_k \cdot \mathbf{v}^{(l)} \, dx = \int\limits_{\omega_k} \operatorname{div} (\theta_k \mathbf{v}^{(l)}) \, dx = \int\limits_{\partial\omega_k} \theta_k \mathbf{v}^{(l)} \cdot \mathbf{n} \, dx = \int\limits_{\sigma(R_k)} \mathbf{v}^{(l)} \cdot \mathbf{n} \, dx = 0,$$

a solution $\widehat{\mathbf{v}}_k^{(l)}$ of problem (6.18) exists and satisfies the estimate

[5] Notice that the embedding $\mathbb{H}_S(\Omega_l) \hookrightarrow L^4_S(\Omega_l)$ is compact.

$$\|\nabla\widehat{\mathbf{v}}_k^{(l)}\|_{L^2(\omega_k)} \quad \le \quad c\|\nabla\theta_k\cdot\mathbf{v}^{(l)}\|_{L^2(\omega_k)}, \tag{6.19}$$

where c is independent of k (see Lemma 6.3). Using the estimate (6.17) and the Poincaré inequality (6.2), from (6.19) we derive the estimate

$$\|\nabla\widehat{\mathbf{v}}_k^{(l)}\|_{L^2(\omega_k)}\le c\|\nabla\theta_k\cdot\mathbf{v}^{(l)}\|_{L^2(\omega_k)}\le\frac{c}{g(R_k)}\|\mathbf{v}^{(l)}\|_{L^2(\omega_k)}\le c\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)}. \tag{6.20}$$

Notice that $\widehat{\mathbf{v}}_k^{(l)}$ is not necessary symmetric, so we symmetrized it as in (4.22). For simplicity we do not change the notation of $\widehat{\mathbf{v}}_k^{(l)}$, i.e. $\widehat{\mathbf{v}}_k^{(l)}$ is symmetric in the following text.

Set $\boldsymbol{\eta}=\mathbf{U}_k^{(l)}$ in (6.8). Then, because $\mathbf{U}_k^{(l)}=0$ in $\Omega\setminus\Omega_{k+1}$ and

$$\int\limits_{\Omega_{k+1}}((\mathbf{v}^{(l)}+\mathbf{A})\cdot\nabla)\mathbf{U}_k^{(l)}\cdot\mathbf{U}_k^{(l)}\,dx=0,$$

we obtain

$$\begin{aligned}
&\nu\int\limits_{\Omega_k}|\nabla\mathbf{v}^{(l)}|^2\,dx=\int\limits_{\omega_k}((\mathbf{v}^{(l)}+\mathbf{A})\cdot\nabla)\,\mathbf{U}_k^{(l)}\cdot(\mathbf{v}^{(l)}-\mathbf{U}_k^{(l)})\,dx\\
&-\nu\int\limits_{\omega_k}\nabla\mathbf{v}^{(l)}:\nabla\mathbf{U}_k^{(l)}\,dx+\int\limits_{\Omega_{k+1}}(\mathbf{v}^{(l)}\cdot\nabla)\,\mathbf{U}_k^{(l)}\cdot\mathbf{A}\,dx\\
&-\nu\int\limits_{\Omega_{k+1}}\nabla\mathbf{A}:\nabla\mathbf{U}_k^{(l)}\,dx+\int\limits_{\Omega_{k+1}}(\mathbf{A}\cdot\nabla)\,\mathbf{U}_k^{(l)}\cdot\mathbf{A}\,dx+\int\limits_{\Omega_{k+1}}\mathbf{f}\cdot\mathbf{U}_k^{(l)}\,dx.
\end{aligned} \tag{6.21}$$

In order to estimate the right hand side of (6.21), we use first the inequalities (6.20), (6.3) and the Poincaré inequality (6.2) to obtain

$$\begin{aligned}
\|\mathbf{v}^{(l)}\|_{L^4(\omega_k)} &\le c\,g^{1/2}(R_k)\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)};\\
\|\mathbf{v}^{(l)}-\mathbf{U}_k^{(l)}\|_{L^4(\omega_k)} &\le \|\mathbf{v}^{(l)}\|_{L^4(\omega_k)}+\|\widehat{\mathbf{v}}_k^{(l)}\|_{L^4(\omega_k)}\\
&\le c\,g^{1/2}(R_k)\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)}+c\,g^{1/2}(R_k)\|\nabla\widehat{\mathbf{v}}_k^{(l)}\|_{L^2(\omega_k)}\\
&\le c\,g^{1/2}(R_k)\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)};\\
\|\nabla\mathbf{U}_k^{(l)}\|_{L^2(\omega_k)} &\le \|\nabla(\theta_k\mathbf{v}^{(l)})\|_{L^2(\omega_k)}+\|\nabla\widehat{\mathbf{v}}_k^{(l)}\|_{L^2(\omega_k)}\\
&\le \|\nabla\theta_k\|_{L^\infty(\omega_k)}\,\|\mathbf{v}^{(l)}\|_{L^2(\omega_k)}+\|\theta_k\|_{L^\infty(\omega_k)}\,\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)}\\
&+ c\,\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)}\le c\,\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)}.
\end{aligned} \tag{6.22}$$

Below we will need the following inequality

$$\int\limits_{\omega_k}|\mathbf{A}|^2|\mathbf{w}|^2dx\le c\,\varepsilon^2\int\limits_{\omega_k}|\nabla\mathbf{w}|^2dx\quad\forall\mathbf{w}\in\mathbb{W}^{1,2}_{loc}(\overline{\Omega}),\ \mathbf{w}=0\text{ on }\partial\Omega, \tag{6.23}$$

which can be proved arguing like for proving Leray-Hopf's inequality.
By using the Hölder inequality, (6.22) and (6.23) we obtain

$$
\begin{aligned}
&\Big|\int\limits_{\omega_k}((\mathbf{v}^{(l)}+\mathbf{A})\cdot\nabla)\mathbf{U}_k^{(l)}\cdot(\mathbf{v}^{(l)}-\mathbf{U}_k^{(l)})\,dx\Big|\\
&\le\Big(\|\mathbf{v}^{(l)}\|_{L^4(\omega_k)}\|\mathbf{v}^{(l)}-\mathbf{U}_k^{(l)}\|_{L^4(\omega_k)}+\|\mathbf{A}\,(\mathbf{v}^{(l)}-\mathbf{U}_k^{(l)})\|_{L^2(\omega_k)}\Big)\|\nabla\mathbf{U}_k^{(l)}\|_{L^2(\omega_k)}\\
&\le c\,g(R_k)\,\|\nabla\mathbf{v}^{(l)}\|^3_{L^2(\omega_k)}+c\,\varepsilon\,\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)}\,\|\nabla(\mathbf{v}^{(l)}-\mathbf{U}_k^{(l)})\|_{L^2(\omega_k)}\\
&\le c\,g(R_k)\,\|\nabla\mathbf{v}^{(l)}\|^3_{L^2(\omega_k)}+c\,\varepsilon\,\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_k)}.
\end{aligned}
\tag{6.24}
$$

We estimate the second term of the equation (6.21) by using the Cauchy-Schwarz inequality and the estimates (6.22):

$$
\nu\Big|\int\limits_{\omega_k}\nabla\mathbf{v}^{(l)}:\nabla\mathbf{U}_k^{(l)}\,dx\Big|\le\nu\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)}\,\|\nabla\mathbf{U}_k^{(l)}\|_{L^2(\omega_k)}\le\nu c\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_k)}. \tag{6.25}
$$

To estimate the third term of (6.21) we use the Leray-Hopf inequality (5.9), the Hölder inequality, (6.22) and (6.23):

$$
\begin{aligned}
&\Big|\int\limits_{\Omega_{k+1}}(\mathbf{v}^{(l)}\cdot\nabla)\mathbf{U}_k^{(l)}\cdot\mathbf{A}\,dx\Big|\\
&\le\Big|\int\limits_{\Omega_k}(\mathbf{v}^{(l)}\cdot\nabla)\mathbf{v}^{(l)}\cdot\mathbf{A}\,dx\Big|+\Big|\int\limits_{\omega_k}(\mathbf{v}^{(l)}\cdot\nabla)\mathbf{U}_k^{(l)}\cdot\mathbf{A}\,dx\Big|\\
&\le c\,\varepsilon\,\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_k)}+\|\nabla\mathbf{U}_k^{(l)}\|_{L^2(\omega_k)}\Big(\int\limits_{\omega_k}|\mathbf{v}^{(l)}|^2|\mathbf{A}|^2\,dx\Big)^{1/2}\\
&\le c\,\varepsilon\Big(\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_k)}+\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_k)}\Big).
\end{aligned}
\tag{6.26}
$$

The last three terms of (6.21) are estimated by using the Hölder inequality, the Cauchy inequality, (6.13), (6.14) and (6.22):

$$
\begin{aligned}
&\nu\Big|\int\limits_{\Omega_{k+1}} \nabla\mathbf{A} : \nabla\mathbf{U}_k^{(l)}\,dx\Big| + \Big|\int\limits_{\Omega_{k+1}} (\mathbf{A}\cdot\nabla)\mathbf{U}_k^{(l)}\cdot\mathbf{A}\,dx\Big| + \Big|\int\limits_{\Omega_{k+1}} \mathbf{f}\cdot\mathbf{U}_k^{(l)}\,dx\Big| \\
&\le c\Big(\|\nabla\mathbf{A}\|_{L^2(\Omega_{k+1})} + \|\mathbf{A}\|^2_{L^4(\Omega_{k+1})} + \|\mathbf{f}\|_{H^*(\Omega_{k+1})}\Big)\|\nabla\mathbf{U}_k^{(l)}\|_{L^2(\Omega_{k+1})} \\
&\le \frac{c}{2\,\mu}\Big(\|\nabla\mathbf{A}\|_{L^2(\Omega_{k+1})}+\|\mathbf{A}\|^2_{L^4(\Omega_{k+1})}+\|\mathbf{f}\|_{H^*(\Omega_{k+1})}\Big)^2 + \frac{c\,\mu}{2}\|\nabla\mathbf{U}_k^{(l)}\|^2_{L^2(\Omega_{k+1})} \qquad (6.27)\\
&\le \frac{2\,c}{\mu}\Big(\|\nabla\mathbf{A}\|^2_{L^2(\Omega_{k+1})} + \|\mathbf{A}\|^4_{L^4(\Omega_{k+1})} + \|\mathbf{f}\|^2_{H^*(\Omega_{k+1})}\Big) + \frac{c\,\mu}{2}\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{k+1})} \\
&\le \frac{c(\mathbf{a},\|\mathbf{f}\|_*)}{\mu}\Big(1 + \int\limits_{R_0}^{R_{k+1}} \frac{dx_1}{g^3(x_1)}\Big) + c\,\frac{\mu}{2}\Big(\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_k)} + \|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_k)}\Big).
\end{aligned}
$$

Therefore, from (6.21), (6.24), (6.26), (6.27) it follows that

$$
\begin{aligned}
&\nu\int\limits_{\Omega_k} |\nabla\mathbf{v}^{(l)}|^2\,dx \le c\,g(R_k)\|\nabla\mathbf{v}^{(l)}\|^3_{L^2(\omega_k)} + c\,\varepsilon\,\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_k)} + c\,\nu\,\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_k)} \\
&+c\,(\varepsilon + \frac{\mu}{2})\Big(\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_k)} + \|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_k)}\Big) + \frac{c(\mathbf{a},\|\mathbf{f}\|_*)}{\mu}\Big(1 + \int\limits_{R_0}^{R_{k+1}} \frac{dx_1}{g^3(x_1)}\Big).
\end{aligned}
$$

For ε and μ sufficiently small, we obtain

$$
\begin{aligned}
&\int\limits_{\Omega_k} |\nabla\mathbf{v}^{(l)}|^2\,dx \le c\,g(R_k)\|\nabla\mathbf{v}^{(l)}\|^3_{L^2(\omega_k)} + c\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_k)} \\
&+c(\mathbf{a},\|\mathbf{f}\|_*)\Big(1 + \int\limits_{R_0}^{R_{k+1}} \frac{dx_1}{g^3(x_1)}\Big).
\end{aligned} \qquad (6.28)
$$

Using the remark 2.3 several times we derive

$$
\begin{aligned}
&\int\limits_{R_k}^{R_{k+1}} \frac{dx_1}{g^3(x_1)} \le \int\limits_{R_k}^{R_{k+1}} \frac{dx_1}{\big(\frac{1}{2}\,g(R_k)\big)^3} = \frac{8\,(R_{k+1}-R_k)}{g^3(R_k)} = \frac{4}{L\,g^2(R_k)}, \\
&\int\limits_{R_{k-1}}^{R_k} \frac{dx_1}{g^3(x_1)} \ge \int\limits_{R_{k-1}}^{R_k} \frac{dx_1}{\big(\frac{3}{2}\,g(R_{k-1})\big)^3} = \frac{8\,(R_k-R_{k-1})}{27\,g^3(R_{k-1})} = \frac{4}{27\,L\,g^2(R_{k-1})}.
\end{aligned}
$$

Since $g(R_k) \ge \dfrac{1}{2}g(R_{k-1})$ we get

$$
\int\limits_{R_{k-1}}^{R_k} \frac{dx_1}{g^3(x_1)} \ge \frac{1}{27\,L\,g^2(R_k)}.
$$

It follows that

$$
\int\limits_{R_k}^{R_{k+1}} \frac{dx_1}{g^3(x_1)} \le \frac{4}{L\,g^2(R_k)} = 27\cdot 4\frac{1}{27\,L\,g^2(R_k)} \le 27\cdot 4\int\limits_{R_{k-1}}^{R_k} \frac{dx_1}{g^3(x_1)}.
$$

Thus we have

$$\int\limits_{R_0}^{R_{k+1}} \frac{dx_1}{g^3(x_1)} \le \int\limits_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)} + \int\limits_{R_k}^{R_{k+1}} \frac{dx_1}{g^3(x_1)} \le 109 \int\limits_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)}$$

and the inequality (6.28) becomes

$$\int\limits_{\Omega_k} |\nabla \mathbf{v}^{(l)}|^2\,dx \le c\,g(R_k)\|\nabla \mathbf{v}^{(l)}\|^3_{L^2(\omega_k)} + c\|\nabla \mathbf{v}^{(l)}\|^2_{L^2(\omega_k)} + c(\mathbf{a},\|\mathbf{f}\|_*)\left(1+\int\limits_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)}\right).$$

Denote $y_k = \int\limits_{\Omega_k} |\nabla \mathbf{v}^{(l)}|^2\,dx$. Since $\int\limits_{\omega_k} = \int\limits_{\Omega_{k+1}} - \int\limits_{\Omega_k}$, we can rewrite the last inequality as

$$y_k \le c_*(y_{k+1}-y_k) + c_{**}g(R_k)(y_{k+1}-y_k)^{3/2} + \frac{1}{2}Q_k, \tag{6.29}$$

where

$$Q_k = 2\,c(\mathbf{a},\|\mathbf{f}\|_*)\left(1+\int\limits_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)}\right). \tag{6.30}$$

We have, using the remark 2.3 again

$$\begin{aligned}
&c_*(Q_{k+1}-Q_k) + c_{**}g(R_k)(Q_{k+1}-Q_k)^{3/2}\\
&\le 2\,c_*\,c(\mathbf{a},\|\mathbf{f}\|_*)\int\limits_{R_k}^{R_{k+1}} \frac{dx_1}{g^3(x_1)} + c_{**}\,g(R_k)\left(c(\mathbf{a},\|\mathbf{f}\|_*)\int\limits_{R_k}^{R_{k+1}} \frac{dx_1}{g^3(x_1)}\right)^{3/2}\\
&\qquad \le c_1\,c(\mathbf{a},\|\mathbf{f}\|_*)\,g^{-2}(R_k) \le c_2\,c(\mathbf{a},\|\mathbf{f}\|_*)\int\limits_{R_{k-1}}^{R_k} \frac{dx_1}{g^3(x_1)}\\
&\qquad \le c(\mathbf{a},\|\mathbf{f}\|_*)\left(1+\int\limits_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)}\right) = \frac{1}{2}Q_k
\end{aligned}$$

for k large enough if $\dfrac{\int\limits_{R_{k-1}}^{R_k} \frac{dx_1}{g^3(x_1)}}{1+\int\limits_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)}} \to 0$ when $k \to +\infty$.

Claim. *Let non negative numbers $y_k,\ k=1,...,N$, satisfy the inequalities $y_{k+1} \ge y_k$ and*

$$y_k \le c_*(y_{k+1}-y_k) + c_{**}\,g(R_k)(y_{k+1}-y_k)^{3/2} + \frac{1}{2}Q_k, \tag{6.31}$$

where c_*, c_{**}, Q_k are non negative numbers such that

$$\frac{1}{2}Q_k \ge c_*(Q_{k+1}-Q_k) + c_{**}\,g(R_k)(Q_{k+1}-Q_k)^{3/2}. \tag{6.32}$$

If $N < +\infty$ and $y_N \le Q_N$ then $y_k \le Q_k \quad \forall k < N$.

Although this claim is proved in [58], for the reader convenience we give the proof which is based on induction. Suppose we have proved that $y_{k+1} \le Q_{k+1}$. If $y_k > Q_k$ then $0 \le y_{k+1} - y_k < Q_{k+1} - Q_k$. Since the function $\tau \to F(\tau) = c_*\tau + c_{**}\, g(R_k)\, \tau^{3/2}$ is increasing we get

$$y_k \le F(y_{k+1} - y_k) + \frac{1}{2}Q_k < F(Q_{k+1} - Q_k) + \frac{1}{2}Q_k \le \frac{1}{2}Q_k + \frac{1}{2}Q_k = Q_k$$

and a contradiction. Thus, $y_k \le Q_k$.

Since Q_k satisfies the condition (6.32), the inequality (6.29) together with (6.15) and the claim above, the estimate

$$y_k = \int\limits_{\Omega_k} |\nabla \mathbf{v}^{(l)}|^2\, dx \le c(\mathbf{a}, \|\mathbf{f}\|_*) \left(1 + \int\limits_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)}\right) \quad \forall k \le l \tag{6.33}$$

holds.

Since for every bounded domain Ω_k, $k > 0$ the embedding $\mathbb{W}_S^{1,2}(\Omega_k) \hookrightarrow \mathbb{L}_S^4(\Omega_k)$ is compact, the estimate (6.33) is sufficient to assure the existence of a subsequence $\{\mathbf{v}^{(l_m)}\}$ which converges weakly in $\mathring{\mathbb{W}}_S^{1,2}(\Omega_k)$ and strongly in $\mathbb{L}_S^4(\Omega_k)$ for any $k > 0$. Such subsequence could be constructed by Cantor diagonal process: we can choose a weakly convergent subsequence $\{\mathbf{v}^{(l_m)}\}$ in $\mathring{W}_S^{1,2}(\Omega_1)$ which converges strongly in $\mathbb{L}_S^4(\Omega_1)$. In the same manner we subtract a subsequence of $\{\mathbf{v}^{(l_m)}\}$ in Ω_2 which we call also $\{\mathbf{v}^{(l_m)}\}$ for the sake of simplicity. Continuing this we can choose a desired subsequence. Taking in integral identity (6.8) an arbitrary test function $\boldsymbol{\eta}$ with a compact support, we can find a number k such that $\operatorname{supp} \boldsymbol{\eta} \subset \Omega_k$ and $\boldsymbol{\eta} \in \mathbb{H}_S(\Omega_k)$. Extending $\boldsymbol{\eta}$ by zero into $\Omega \setminus \Omega_k$, and considering all integrals in (6.8) as integrals over Ω, we can pass in (6.8) to a limit as $l_m \to +\infty$. As a result we get for the limit vector function $\mathbf{v}$ the integral identity (2.5). Therefore, $\mathbf{u} = \mathbf{A} + \mathbf{v}$ is a weak solution of problem (1.1). The estimate (6.15) for $\mathbf{v}$ follows from (6.33). Since for $\mathbf{A}$ the analogous to (6.15) is also valid, we obtain (6.15) for the sum $\mathbf{u} = \mathbf{A} + \mathbf{v}$.

Remark 6.2. If the norms $\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}$ and $\|\mathbf{f}\|_*$ are sufficiently small, it can be proved using the methods proposed in [36] and [58] that the weak solution $\mathbf{u}$ is unique in a class of functions with the Dirichlet integral growing "not too fast".

Remark 6.3. If the D_j's are channel-like outlets and $|\mathbb{F}|$ is sufficiently small, it can be proved using the methods from [36] and [58] that the weak solution $\mathbf{u}$ tends to the Poiseuille flow as $x_1 \to +\infty$. In this sense our result extends the result obtained by H. Morimoto and H. Fujita in [40], [41].

Remark 6.4. If we remove the symmetry assumptions the proof of the existence results in this section works for non symmetric domains, i.e. we have also showed the following theorem.

Theorem 6.3. *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain containing finitely many outlets to infinity. Assume that the boundary value $\boldsymbol{a} \in \mathbb{W}^{1/2,2}(\partial\Omega)$ has a compact support, the external force* $\mathbf{f}$ *is a vector field such that for every k the integral $\int\limits_{\Omega_k} \mathbf{f}\cdot\boldsymbol{\eta}\,dx$ defines a bounded functional on $\mathbb{H}(\Omega_k)$. Then problem (2.2) admits at least one weak solution $\mathbf{u} = \mathbf{A} + \mathbf{v}$. Furthermore, if $\int\limits_{R_{j,0}}^{+\infty} \frac{dz_1^{(j)}}{g_j^3(z_1^{(j)})} < +\infty, \quad j = 1, \dots, N$, then the weak solution* $\mathbf{u}$ *satisfies the estimate*

$$\int\limits_{\Omega} |\nabla \boldsymbol{u}|^2\,dx \le c(\mathbf{a}, \|\mathbf{f}\|_*)\left(1 + \sum_{j=1}^{N} \int\limits_{R_{j,0}}^{+\infty} \frac{dz_1^{(j)}}{g_j^3(z_1^{(j)})}\right) \tag{6.34}$$

while if there is a number $j \in \{1, \dots, N\}$ such that

$$\int\limits_{R_{j,0}}^{+\infty} \frac{dz_1^{(j)}}{g_j^3(z_1^{(j)})} = +\infty, \text{ and } \frac{\int\limits_{R_{j,k-1}}^{R_{j,k}} \frac{dx_1}{g^3(z_1^{(j)})}}{1 + \int\limits_{R_{j,0}}^{R_{j,k}} \frac{dz_1^{(j)}}{g^3(z_1^{(j)})}} \to 0 \text{ when } k \to +\infty, \text{ then } \mathbf{u} \text{ satisfies}$$

$$\int\limits_{\Omega_k} |\nabla \boldsymbol{u}|^2\,dx \le c(\mathbf{a}, \|\mathbf{f}\|_*)\left(1 + \sum_{j=1}^{N} \int\limits_{R_{j,0}}^{R_{j,k}} \frac{dz_1^{(j)}}{g_j^3(z_1^{(j)})}\right), \tag{6.35}$$

where

$$\|\mathbf{f}\|_* = \sup_{k\ge 1}\left(\Big(1 + \sum_{j=1}^{N} \int\limits_{R_{j,0}}^{R_{j,k}} \frac{dz_1^{(j)}}{g_j^3(z_1^{(j)})}\Big)^{-1/2} \cdot \|\mathbf{f}\|_{H^*(\Omega_k)}\right),$$

$$\|\mathbf{f}\|_{H^*(\Omega_k)} = \sup_{\boldsymbol{\eta}\in J_0^\infty(\Omega_k)} \frac{\left|\int\limits_{\Omega_k} \mathbf{f}\cdot\boldsymbol{\eta}\,dx\right|}{\|\boldsymbol{\eta}\|_{D(\Omega_k)}},$$

$c(\mathbf{a}, \|\mathbf{f}\|_*) = c\left(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{f}\|_*^2\right)$ *and c is independent of k.*

7 Conclusion

We have constructed a solenoidal extension of the boundary value satisfying the Leray-Hopf inequalities in the case where the flux on each connected component of the inner boundary equals to zero. There is no restriction on the flux on outer boundary. We have showed that the steady Navier-Stokes equations possess at least one weak solution.

If the given domain and all data are symmetric with respect to the x_1-axis. We have constructed a symmetric extension satisfying the Leray-Hopf inequalities hence there exists a symmetric weak solution to the Navier-Stokes equations. In the symmetric case we do not impose any restriction on the flux over the boundary.

Chapter 3

Navier-Stokes equations in a punctured periodic domain

We study in this chapter fluid flow around an obstacle. It is a challenging and interesting problem in fluid mechanics, and has been the subject of much experimental and numerical investigation (see, among others, [3, 9, 17, 19, 48, 56, 63, 65]).

The mathematical analysis of the influence of an obstacle on the behaviour of the flow when the size of the obstacle is small when compared to that of the reference spatial scale has recently received increased attention. The case of a single obstacle in a two-dimensional ideal flow was analysed by Iftimie, Lopes Filho, & Nussenzveig Lopes [21]; then Iftimie et al. [22] and Iftimie & Kelliher [20] considered the viscous case, Lopes Filho [38] treated bounded domains with several holes, Lacave [31, 32] considered obstacles that shrink to a curve, and moving obstacles were treated by Dahsti & Robinson [8] and Silvestre & Takahashi [55]. For problems in exterior domains (i.e. extending to infinity) the flow is usually assumed to vanish at infinity, although the case of flows constant at infinity has been recently considered by Lopes Filho, Nguyen, & Nussenzveig Lopes [39]. A related 'small body' problem was considered by Robinson [53], who treated a simplified model of combustion in which physical particles were replaced by diffuse but compact regions of influence in the flow.

Here we are interested in the vanishing obstacle problem in a 2D periodic domain with a particularly simple geometry. More precisely, we are concerned with periodic flows on the punctured domain

$$\Omega_r = (-L, L)^2 \setminus D_r,\ L > 0,$$

where $D_r = B(0, r)$ is the disc of radius r centred at the origin, and we study the behaviour of the solutions of various models when the radius r of the disc tends to zero. Throughout this chapter we refer to the excised disc D_r as the 'obstacle' in keeping with the ultimate application to problems of fluid flow.

First we consider the Poisson equation as a model problem, prior to treating to stationary Stokes and Navier–Stokes problems, which have the added component of incompressibility. Thus our initial aim (in Section 2) will be to determine the asymptotic behaviour of the solution of the following problem when $r \to 0$:

$$-\Delta u_r = f \quad \text{in } \Omega_r, \qquad u_r \text{ periodic}, \qquad u_r = 0 \text{ on } \partial D_r. \tag{0.1}$$

While this problem has a solution for any $f \in L^2(\Omega_r)$, the limiting problem,

$$-\Delta u = f \quad \text{in } \Omega, \qquad u \text{ periodic},$$

only has a solution when

$$\int\limits_{\Omega} f = 0. \tag{0.2}$$

We will show that when (0.2) holds then the solutions of (0.1) are uniformly bounded in r in the sense that

$$\int\limits_{\Omega_r} |\nabla u_r|^2 + \int\limits_{\Omega_r} \left| u_r - \fint_{\Omega} u_r \right|^2$$

is uniformly bounded, where $\fint_\Omega u = |\Omega|^{-1} \int_\Omega u$ denotes the average of u over Ω (note that this is the whole domain and not just Ω_r). This is enough to show that

$$u_r - \fint_{\Omega} u_r \to u$$

in $H^1(\Omega)$ and that u satisfies the limiting equation. If (0.2) does not hold then the limiting problem has no solution, and in this case it follows that $\|u_r\|_{H^1}$ is unbounded as $r \to 0$.

We remark here, and will return to this later, that we have been unable to obtain a uniform bound on $\fint_\Omega u_r$, since the constant in the Poincaré inequality available on Ω_r degrades as $r \to 0$ (see Lemma 1.1).

In Section 3 we obtain similar results for the Stokes problem

$$\begin{cases} -\Delta \mathbf{u}_r + \nabla p_r = \mathbf{f} \quad \text{in } \Omega_r, \\ \text{div } \mathbf{u}_r = 0, \\ \mathbf{u}_r \text{ periodic}, \\ \mathbf{u}_r = 0 \text{ on } \partial D_r. \end{cases}$$

The main change from the case of the pure Laplacian is that we now have to deal with divergence-free vector-valued functions. The key technical result that allows us to do this is a method for approximating divergence-free periodic functions defined on the whole of Ω by a sequence of divergence-free functions that satisfy the zero boundary condition on D_r (Lemma 2.1). Once again, we require that $\int_\Omega \mathbf{f} = 0$. As before, we can find uniform estimates sufficient to show that $\mathbf{u}_r - \fint_\Omega \mathbf{u}_r$ converges to a solution of the limiting problem, but we are unable to bound the average of $\mathbf{u}_r$ over Ω.

It would seem that the next natural step would be to consider the stationary Navier–Stokes equations in Ω_r,

$$-\Delta \mathbf{u}_r + (\mathbf{u}_r \cdot \nabla)\mathbf{u}_r + \nabla p_r = \mathbf{f}, \qquad \nabla \cdot \mathbf{u}_r = 0. \tag{0.3}$$

However, while in the linear problems considered so far bounds on $\mathbf{u}_r - \fint_\Omega \mathbf{u}_r$ were sufficient to pass to the limit, this is not the case here. Informally, if we set $\langle \mathbf{u}_r \rangle = \fint_\Omega \mathbf{u}_r$ and consider the equation for $\tilde{\mathbf{u}}_r = \mathbf{u}_r - \langle \mathbf{u}_r \rangle$ then we obtain

$$-\Delta \tilde{\mathbf{u}}_r + (\tilde{\mathbf{u}}_r \cdot \nabla)\tilde{\mathbf{u}}_r + (\langle \mathbf{u}_r \rangle \cdot \nabla)\tilde{\mathbf{u}}_r + \nabla p_r = \mathbf{f},$$

which contains the additional term $(\langle \mathbf{u}_r \rangle \cdot \nabla)\tilde{\mathbf{u}}_r$. A uniform bound on $\langle \mathbf{u}_r \rangle$ would enable us to pass to the limit in this term, but we do not currently have such a bound.

An additional factor that makes this problem different in character from the others we consider here is that there is no known general uniqueness result for solutions of (0.3), even on the entire periodic domain. As such, it is perhaps more natural to consider a perturbation problem (given a solution of the equation on Ω, investigate the existence of nearby solutions for r small) than as a limiting problem; or to treat a restricted setting in which uniqueness results are available (when $\mathbf{f}$ is small in an appropriate sense). For more discussion of this stationary problem we refer to the classical work of Ladyzhenskaya [33] and Temam [60, 61].

The main technical difficulty is the lack of a uniform L^2 bound for the weak solution. It is worth to notice that there is no restriction on the average of the solution for the expected limiting problem, i.e., for the stationary Navier–Stokes equations with periodic boundary conditions in Ω. Moreover, the constant in the Poincaré inequality degrades with $r \to 0$ (see Lemma 1.1), so we can not employ it to obtain the L^2 bound from the estimate on the gradient of the solution.

1 Poisson Equation

In this section we discuss the asymptotic behaviour of weak solutions for the Poisson problem

$$\begin{cases} -\Delta u_r = f \ \text{ in } \Omega_r, \\ u_r \text{ periodic}, \\ u_r = 0 \text{ on } \partial D_r. \end{cases}$$

Let us introduce some notation. Set $\Omega_0 = (-L, L)^2 = \Omega$ and define the function spaces

$$H^1_{\mathrm{per}}(\Omega) = \{v \in H^1(\Omega) : \text{ periodic}\}$$

and, for $r > 0$,

$$H^1_{\mathrm{per}}(\Omega_r) = \text{ the closure of } C^1_{\mathrm{per}}(\overline{\Omega}_r) \text{ in } H^1(\Omega_r)$$

and

$$V_{0,r} = \{v \in H^1_{\mathrm{per}}(\Omega_r) : \ v = 0 \text{ on } \partial D_r\}.$$

Note that any function in $V_{0,r}$ can be extended by zero inside D_r to give a function in $H^1_{\mathrm{per}}(\Omega)$; this observation is fundamental to our analysis.

The vanishing obstacle problem for the Poisson equation

$$-\Delta u_r = f \ \text{ in } \Omega_r, \qquad u_r \in V_{0,r}, \tag{1.1}$$

consists in determining the asymptotic behaviour of the solution u_r when r tends to 0.

The precise statement of our first convergence result is as follows.

Theorem 1.4. *Let $f \in L^2(\Omega)$. For every $r > 0$ there exists a unique solution $u_r \in V_{0,r}$ of the problem*

$$\int_{\Omega_r} \nabla u_r \cdot \nabla v = \int_{\Omega_r} f v \qquad \text{for all } v \in V_{0,r}. \tag{1.2}$$

Moreover

a) if $\int_\Omega f = 0$ then as $r \to 0$

$$u_r - \frac{1}{|\Omega|}\int_\Omega u_r \to u_0 \qquad \text{and} \qquad \nabla u_r \to \nabla u_0,$$

where the limits are taken in $L^2(\Omega)$ and $u_0 \in H^1_{\mathrm{per}}(\Omega)$ is the unique solution of the problem

$$\int_\Omega \nabla u_0 \cdot \nabla v = \int_\Omega f v \qquad \text{for all } v \in H^1_{\mathrm{per}}(\Omega) \tag{1.3}$$

that satisfies $\int_\Omega u_0 = 0$.

b) If $\int_\Omega f \neq 0$ then $\|\nabla u_r\|_{L^2}$ is unbounded as $r \to 0$.

A few comments are in order.

Note that one can use $v = 1$ as a test function in (1.3), from which it follows immediately that there can be no solution of the limiting problem unless

$$\int_\Omega f = 0.$$

Observe that we do not have convergence of u_r itself in $L^2(\Omega)$. The main reason for this is that the constant in the Poincaré inequality for the punctured domain Ω_r degrades as $r \to 0$. We first recall the classical Poincaré inequality: there exists a constant $C > 0$ such that for any $v \in H^1_{\text{per}}(\Omega)$

$$\left\| v - \fint v \right\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)}, \tag{1.4}$$

where

$$\fint v = \frac{1}{|\Omega|} \int_\Omega v.$$

Notice that inequality (1.4) is still valid for functions in $v \in V_{0,r}$, and in particular the constant does not depend on r. However, without subtraction of the average we have only the following estimate.

Lemma 1.1. *Let $r < (2 - \sqrt{2})L$. Then for all $v \in V_{0,r}$*

$$\|v\|_{L^2(\Omega_r)} \leq c(-\log r) \|\nabla v\|_{L^2(\Omega_r)}.$$

Proof. We assume that $v \in C^1_{\text{per}}(\Omega_r)$ with $v = 0$ on ∂D_r, then $v \in V_{0,r}$. We extend v periodically outside Ω_r, the assumption that $r < (2 - \sqrt{2})L$ meaning that any x with $|x| \leq \sqrt{2}L$ in the extended domain does not lie within one of the additional 'holes', see Figure 3.1.

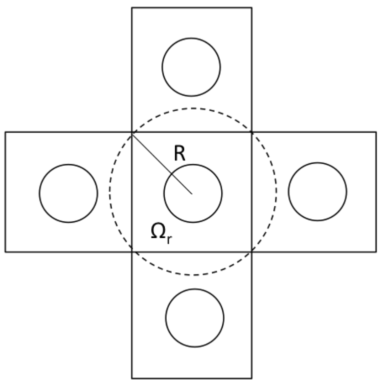


Figure 3.1: Periodic extension of the domain Ω_r used in the proof of Lemma 1.1

At $x = \rho\hat{x}$ (where $\hat{x} = x/|x|$), we can write

$$v(x) = v(\rho\hat{x}) - v(r\hat{x}) = \int_r^\rho \frac{\mathrm{d}}{\mathrm{d}s} v(s\hat{x}) \, \mathrm{d}s \leq \int_r^\rho |\nabla v(s\hat{x})| \, \mathrm{d}s.$$

Then, since $B(0,\sqrt{2}L)\supset\Omega_r$, setting $R=\sqrt{2}L$ we have

$$
\begin{aligned}
\int_{\Omega_r}|v(x)|^2 &\le \int_0^{2\pi}\int_r^R \rho|v(\rho\hat{x})|^2\,\mathrm{d}\rho\,\mathrm{d}\theta\\
&\le \int_0^{2\pi}\int_r^R \rho\left(\int_r^\rho |\nabla v(s\hat{x})|\,\mathrm{d}s\right)^2\mathrm{d}\rho\,\mathrm{d}\theta\\
&\le \int_0^{2\pi}\int_r^R \rho\left(\int_r^\rho s^{-1}\,\mathrm{d}s\right)\left(\int_r^\rho s|\nabla v(s\hat{x})|^2\,\mathrm{d}s\right)\mathrm{d}\rho\,\mathrm{d}\theta\\
&\le \int_0^{2\pi}\int_r^R \rho\log(\rho/r)\left(\int_r^\rho s|\nabla v(s\hat{x})|^2\,\mathrm{d}s\right)\mathrm{d}\rho\,\mathrm{d}\theta\\
&\le \left(\int_r^R \rho\log(\rho/r)\,\mathrm{d}\rho\right)\left(\int_{B(0,R)}|\nabla v|^2\,\mathrm{d}x\right),\\
&\le c(-\log r)\|\nabla v\|^2_{L^2(\Omega_r)},
\end{aligned}
$$

using the fact that $\int_{B(0,R)}|\nabla u|^2\le 2\int_{\Omega_r}|\nabla u|^2$ since we have extended u periodically outside Ω_r. □

We note that the fact that the constant in Lemma 1.1 is not independent of r is not merely an artefact of our method of proof: while it may be possible to improve the dependence on r, one cannot remove it. Indeed, consider the family of functions u_r defined on Ω_r by

$$u_r(x)=\log(1+\log(\rho/r))$$

where ρ is distance of x from the origin. This defines a function in $V_{0,r}$, since its values on the boundary of Ω agree on opposite faces.

Now, certainly

$$
\begin{aligned}
\|u_r\|^2_{L^2(\Omega_r)}\ge \int_{r\le|x|\le L}|u_r(x)|^2\,\mathrm{d}x &= 2\pi\int_r^L \rho(\log(1+\log(\rho/r)))^2\,\mathrm{d}\rho\\
&= 2\pi r^2\int_1^{L/r} s(\log(1+\log(s))^2\,\mathrm{d}s\\
&\ge 2\pi r^2\int_{L/2r}^{L/r} s(\log(1+\log s))^2\,\mathrm{d}s\\
&\ge 2\pi r^2(L/2r)^2\log(1+\log(L/2r))^2\\
&= \frac{L^2\pi}{2}\log(1+\log(L/2r))^2,
\end{aligned}
$$

which is unbounded as $r \to 0$. However,

$$\partial_\rho u_r = \frac{1}{1+\log(\rho/r)}\frac{1}{\rho}$$

and so

$$\|\nabla u_r\|_{L^2(\Omega_r)} \leq \int\limits_{r\leq|x|\leq\sqrt{2}L} |\partial_\rho u_r|^2 \,\mathrm{d}x = 2\pi \int\limits_r^{\sqrt{2}L} \frac{1}{(1+\log(\rho/r))^2}\frac{1}{\rho}\,\mathrm{d}\rho$$
$$\leq 2\pi \int\limits_1^\infty \frac{1}{s(1+\log s)^2}\,\mathrm{d}s < \infty.$$

We now state a preliminary lemma on approximation of functions in $H^1_{\mathrm{per}}(\Omega)$ by functions in $V_{0,r}$ which will be used to pass to the limit.

Lemma 1.2. *Given $v \in H^1_{\mathrm{per}}(\Omega)$ there exists a "sequence" $v_\epsilon \in V_{0,\epsilon}$ such that*

$$v_\epsilon \rightharpoonup v \quad \text{in} \quad H^1(\Omega) \qquad \text{as} \qquad \epsilon \to 0.$$

Proof. We first assume that $v \in H^1_{\mathrm{per}} \cap L^\infty(\Omega)$. Let

$$\phi_\epsilon = \min(1, \epsilon^{-1}\mathrm{dist}(x, D_\epsilon)).$$

Observe that $\phi_\epsilon = 0$ for $x \in D_\epsilon$, $\phi_\epsilon = 1$ when $\mathrm{dist}(x, D_\epsilon) \geq \epsilon$, and $|\nabla\phi_\epsilon| = 1/\epsilon$ for $0 < \mathrm{dist}(x, D_\epsilon) < \epsilon$. It follows that $v\phi_\epsilon \in V_{0,\epsilon}$ with

$$|v\phi_\epsilon|^2 \leq |v|^2$$

and

$$|\nabla(v\phi_\epsilon)|^2 = |v\nabla\phi_\epsilon + \phi_\epsilon\nabla v|^2$$
$$\leq 2|v|^2|\nabla\phi_\epsilon|^2 + 2|\nabla v|^2.$$

Since v is bounded and

$$\int\limits_\Omega |\nabla\phi_\epsilon|^2 = \int\limits_{D_{2\epsilon}\setminus D_\epsilon} \epsilon^{-2} = 3\pi,$$

it follows that $v\phi_\epsilon$ is bounded in $H^1(\Omega)$. So we can extract a "subsequence" that converges weakly to a function ω in $H^1(\Omega)$. Since $v\phi_\epsilon \to v$ in $L^2(\Omega)$ we conclude that $\omega = v$.

Finally, given $v \in H^1_{\mathrm{per}}(\Omega)$, note that $v_n = \max(-n, \min(v, n)) \in H^1_{\mathrm{per}} \cap L^\infty(\Omega)$ converges to v in $H^1(\Omega)$ as $n \to \infty$. This allows us to deduce the existence of the required sequence using a diagonal argument. $\square$

We remark that we have shown that $\cup_{\epsilon>0} V_{0,\epsilon}$ is dense in $H^1_{\mathrm{per}}(\Omega)$ in the weak topology. But $\cup_{\epsilon>0} V_{0,\epsilon}$ is a convex set (in fact a vector space) and thus the weak closure is equal to the strong closure and hence $\cup_{\epsilon>0} V_{0,\epsilon}$ is dense in H^1_{per} for the strong topology.

We are now in a position to prove our first convergence result.

Proof (Theorem 1.4). For fixed $r>0$, the existence and uniqueness of u_r follow from the Lax–Milgram Theorem and Lemma 1.1.

We consider the cases when $\int_\Omega f = 0$ and $\int_\Omega f \neq 0$ separately.

a) Assume that $\int_\Omega f = 0$. We first obtain an estimate for the solution u_r. By taking $v = u_r$ in (1.2) and using the Poincaré inequality (1.4) one has

$$\begin{aligned}
\|\nabla u_r\|^2_{L^2} = \int_\Omega |\nabla u_r|^2 &= \int_\Omega f u_r \\
&= \int_\Omega f\left(u_r - \fint_\Omega u_r\right) \\
&\leq \|f\|_{L^2} \left\| u_r - \fint_\Omega u_r \right\|_{L^2} \leq C\|f\|_{L^2}\|\nabla u_r\|_{L^2},
\end{aligned}$$

from which it follows that

$$\|\nabla u_r\|_{L^2} \leq C\|f\|_{L^2}, \tag{1.5}$$

with a constant $C>0$ independent on r.

Next, define

$$\tilde{u}_r = u_r - \fint_\Omega u_r.$$

Then from the bound (1.5) and the Poincaré inequality (1.4), $\|\tilde{u}_r\|_{H^1(\Omega_r)}$ is uniformly bounded.

It follows that, up to the extraction of a "subsequence", $\nabla u_r = \nabla \tilde{u}_r \rightharpoonup \nabla u_0$ and $\tilde{u}_r \to u_0$ in $L^2(\Omega)$. Note that

$$\int_\Omega u_0 = \lim_{r\to 0} \int_\Omega \tilde{u}_r = \lim_{r\to 0} \int_\Omega \left(u_r - \fint_\Omega u_r\right) = 0. \tag{1.6}$$

Now, we pass to the limit in the weak formulation (1.2). Fix $r_0>0$ and observe that for $r<r_0$ one has $V_{0,r_0} \subset V_{0,r}$. Thus,

$$\int_\Omega \nabla u_r \cdot \nabla v = \int_\Omega f v \qquad \text{for all } v \in V_{0,r_0}.$$

The weak convergence of ∇u_r to ∇u_0 in L^2 allows us to pass to the limit and obtain

$$\int_\Omega \nabla u_0 \cdot \nabla v = \int_\Omega f v \qquad \text{for all } v \in V_{0,r_0}. \tag{1.7}$$

From Lemma 1.2, given $v \in H^1_{\text{per}}(\Omega)$ there exists a "sequence" of test functions $v_\epsilon \in V_{0,\epsilon}$ such that $v_\epsilon \rightharpoonup v$ in $H^1(\Omega)$. Thus, for $\epsilon \le r_0$, one has

$$\int_\Omega \nabla u_0 \cdot \nabla v_\epsilon = \int_\Omega f v_\epsilon.$$

Passing to the limit as $\epsilon \to 0$, it follows that

$$\int_\Omega \nabla u_0 \cdot \nabla v = \int_\Omega f v \qquad \text{for all } v \in H^1_{\text{per}}(\Omega),$$

as claimed.

Since the limiting problem has a unique solution when one imposes the zero average condition, it follows that all convergent subsequences must have the same limit. As a consequence, the original sequence converges without the need to extract a subsequence.

It remains to show that in fact $\nabla u_r \to \nabla u_0$ in $L^2(\Omega)$ as $r \to 0$. To this end we show that $\|\nabla u_r\|^2_{L^2} \to \|\nabla u_0\|^2_{L^2}$. Since $u_r - \fint_\Omega u_r \to u_0$ in $L^2(\Omega)$,

$$\int_{\Omega_r} |\nabla u_r|^2 = \int_{\Omega_r} f u_r = \int_\Omega f u_r = \int_\Omega f\left(u_r - \fint_\Omega u_r\right) \to \int_\Omega f u_0.$$

However, from (1.3) we have

$$\int_\Omega |\nabla u_0|^2 = \int_\Omega f u_0,$$

which implies that

$$\int_\Omega |\nabla u_r|^2 \to \int_\Omega |\nabla u_0|^2.$$

Coupled with the weak convergence this norm convergence implies strong convergence of ∇u_r to ∇u_0 in $L^2(\Omega)$.

b) Assume that $\int_\Omega f \neq 0$. We note here that if $\int_\Omega f \neq 0$ and one assumes a uniform bound on $\|\nabla u_r\|_{L^2}$, then one can follow the above argument (apart from obtaining the zero average condition (1.6)) to show that there is a solution of the limiting problem. But as remarked after the statement of Theorem 1.4, there can be no such solution. It follows that in this case $\|\nabla u_r\|_{L^2}$ cannot be uniformly bounded as $r \to 0$. □

We note that in fact $\|\nabla u_r\|_{L^2}$ increases as r decreases. Indeed, note that if $r' < r$ then $V_{0,r} \subset V_{0,r'}$. So we can take $v = u_r$ in both formulations

$$\int_{\Omega_r} \nabla u_r \cdot \nabla v = \int_{\Omega_r} f v \quad \text{and} \quad \int_{\Omega_{r'}} \nabla u_{r'} \cdot \nabla v = \int_{\Omega_{r'}} f v$$

to obtain

$$\int_{\Omega_r} |\nabla u_r|^2 = \int_{\Omega_r} f u_r \quad \text{and} \quad \int_{\Omega_{r'}} \nabla u_{r'} \cdot \nabla u_r = \int_{\Omega_{r'}} f u_r = \int_{\Omega_r} f u_r.$$

Thus

$$\int_{\Omega_r} |\nabla u_r|^2 = \int_{\Omega_{r'}} \nabla u_{r'} \cdot \nabla u_r$$

whence

$$\|\nabla u_r\|^2_{L^2(\Omega_r)} \leq \|\nabla u_{r'}\|_{L^2(\Omega_{r'})} \|\nabla u_r\|_{L^2(\Omega_r)},$$

i.e.

$$\|\nabla u_r\|_{L^2(\Omega_r)} \leq \|\nabla u_{r'}\|_{L^2(\Omega_{r'})}.$$

1.1 Failure of 'uniform elliptic regularity'

We now make the following observation, based on our convergence result, which strongly indicates the possibility that there may be no estimate for second derivatives in this punctured Laplace problem.

Indeed, suppose that there exists a constant $C > 0$, independent of r, such that any solution of

$$-\Delta u_r = f, \qquad u_r|_{\partial D_r} = 0,$$

satisfies

$$\|D^2 u_r\|_{L^2(\Omega_r)} \leq C \|f\|_{L^2(\Omega_r)}.$$

Were this the case, interpolation would imply that u_r is continuous on Ω, and that

$$u_r - ⨍_\Omega u_r \to u_0 \qquad \text{uniformly on } \Omega;$$

in particular, since $u_r = 0$ on ∂D_r, we would have

$$u_0(0) = \lim_{r \to 0} ⨍_\Omega u_r.$$

In other words, a uniform elliptic estimate would imply a uniform estimate on $⨍_\Omega u_r$, and we have not been able to obtain such an estimate.

While this is not a proof that such a uniform elliptic estimate is impossible, we now give an explicit example in a slightly different geometry for which uniform elliptic regularity definitely fails. We consider the same problem in an annulus ('punctured disc')

$$\Omega_r = B(0,2) \setminus B(0,\epsilon),$$

with Dirichlet conditions on the inner and outer boundary. We solve the Poisson equation in plane polar co-ordinates for radially symmetric solutions, using $'$ for $\mathrm{d}/\mathrm{d}r$:

$$\frac{1}{r}(ru')' = f(r) \qquad u(\epsilon) = 0, \qquad u(2) = 0.$$

We take $f = 1 - (3r/4)$ so that $\int_\Omega f \,\mathrm{d}x = \int_0^{2\pi} \int_0^2 r f(r) \,\mathrm{d}r \,\mathrm{d}\theta = 0$.

Then

$$(ru')' = r - \frac{3r^2}{4} \qquad \Rightarrow \qquad ru'(r) = \frac{r^2}{2} - \frac{r^3}{4} + C$$

and so

$$u'(r) = \frac{r}{2} - \frac{r^2}{4} + \frac{C}{r}.$$

Integrating again we obtain

$$u(r) = \frac{r^2}{4} - \frac{r^3}{12} - \frac{\epsilon^2}{4} + \frac{\epsilon^3}{12} + C\log(r/\epsilon),$$

and the boundary condition at $r = 2$ implies that

$$C = \frac{1}{\log(2/\epsilon)}\left[-\frac{1}{3} + \frac{\epsilon^2}{4} - \frac{\epsilon^3}{12}\right].$$

Rewrite the governing equation as

$$u'' + \frac{1}{r}u' = f.$$

Then $\|u''\|_{L^2}$ is bounded by $\|f\|_{L^2} + \|r^{-1}u'\|_{L^2}$. So consider

$$\frac{u'(r)}{r} = \frac{1}{2} - \frac{r}{2} - \frac{C}{r^2}.$$

As the first two terms are in L^2, we need only consider the final term. Noting that

$$\|r^{-1}u'\|_{L^2}^2 = 2\pi C^2 \int_\epsilon^2 r(r^{-1}u')^2 \sim 2\pi C^2 \int_\epsilon^2 \frac{1}{r^3} \sim 2\pi C^2 \epsilon^{-2},$$

so $\|u\|_{\dot{H}^2} \sim \epsilon^{-1}(-\log\epsilon)^{-1}$ with log corrections.

One can find a similar example in the three-dimensional case, namely $f(r) = 1 - 5r^2/3$ on the spherical shell between $r = \epsilon$ and $r = 1$.

The lack of such a bound unfortunately appears to invalidate the arguments treating a moving disc in [8] and a moving sphere in [55].

2 The Stokes equations

In this section we extend the results of the previous section to the Stokes problem

$$-\Delta\mathbf{u}_r + \nabla p_r = \mathbf{f} \text{ in } \Omega_r, \qquad \mathbf{u}_r|_{\partial D_r} = 0, \qquad \operatorname{div}\mathbf{u}_r = 0, \quad \mathbf{u}_r \text{ periodic}.$$

First we introduce the required spaces of vector fields. Given any space of scalar functions X we write $\mathbb{X}$ for the two-component space $X \times X$. Define for $r \geq 0$

$$\mathbb{H}^1_{\text{per}}(\Omega_r) = \text{ the closure of } \mathbb{C}^1_{\text{per}}(\overline{\Omega}_r) \text{ in } \mathbb{H}^1(\Omega_r),$$

$$\mathbb{H}^1_{\text{per},\sigma}(\Omega_r) = \{\mathbf{v} \in \mathbb{H}^1_{\text{per}}(\Omega_r) : \ \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_r\},$$

$$\mathbb{V}_{0,r} = \{\mathbf{v} \in \mathbb{H}^1_{\text{per}}(\Omega_r) : \ \mathbf{v} = 0 \text{ on } \partial D_r\},$$

and

$$\mathbb{V}_{0,r,\sigma} = \{\mathbf{v} \in \mathbb{H}^1_{\text{per},\sigma}(\Omega_r) : \ \mathbf{v} = 0 \text{ on } \partial D_r\}.$$

We observe that any function belonging to $\mathbb{V}_{0,r}$ or $\mathbb{V}_{0,r,\sigma}$ can be extended by zero inside of D_r to give a function in $\mathbb{H}^1_{\text{per}}(\Omega)$ or $\mathbb{H}^1_{\text{per},\sigma}(\Omega)$, respectively.

We will determine the asymptotic behaviour of weak solutions to the following Stokes problem when $r \to 0$:

$$-\Delta \mathbf{u}_r + \nabla p_r = \mathbf{f} \ \text{ in } \Omega_r, \qquad \mathbf{u}_r \in \mathbb{V}_{0,r,\sigma}.$$

Our second convergence result is as follows.

Theorem 2.5. *Let $\mathbf{f} \in \mathbb{L}^2(\Omega)$. For every $r > 0$ there exists a unique solution $\mathbf{u}_r \in \mathbb{V}_{0,r,\sigma}$ of the problem*

$$\int\limits_{\Omega_r} \nabla \mathbf{u}_r : \nabla \mathbf{v} = \int\limits_{\Omega_r} \mathbf{f} \cdot \mathbf{v} \qquad \text{for all } \mathbf{v} \in \mathbb{V}_{0,r,\sigma}. \tag{2.1}$$

Moreover

a) if $\int_\Omega \mathbf{f} = 0$ then as $r \to 0$

$$\mathbf{u}_r - \frac{1}{|\Omega|} \int\limits_{\Omega} \mathbf{u}_r \to \mathbf{u}_0 \qquad \text{and} \qquad \nabla \mathbf{u}_r \to \nabla \mathbf{u}_0,$$

where the limits are taken in $\mathbb{L}^2(\Omega)$ and $\mathbf{u}_0 \in \mathbb{H}^1_{\text{per},\sigma}(\Omega)$ is the unique solution of the problem

$$\int\limits_{\Omega} \nabla \mathbf{u}_0 : \nabla \mathbf{v} = \int\limits_{\Omega} \mathbf{f}\mathbf{v} \qquad \text{for all } \mathbf{v} \in \mathbb{H}^1_{\text{per},\sigma}(\Omega) \tag{2.2}$$

that satisfies $\int_\Omega \mathbf{u}_0 = 0$;

b) if $\int_\Omega \mathbf{f} \neq 0$ then $\|\nabla \mathbf{u}_r\|_{\mathbb{L}^2}$ is unbounded as $r \to 0$.

Note that if we set $\mathbf{v} = (1, 0)$ and $\mathbf{v} = (0, 1)$ as test functions in (2.2), then one can see immediately that for

$$\int_{\Omega} \mathbf{f} \neq 0$$

a solution cannot exist.

The only difference from the Poisson problem is that we now have to approximate functions in $\mathbb{H}^1_{\text{per},\sigma}(\Omega)$ by functions in $\mathbb{V}_{0,r,\sigma}$, i.e. we must incorporate the divergence-free condition. If we have such approximating functions then we can use the same argument as before to show convergence of solutions to those of the limiting problem. Indeed, the Poincaré inequalities work the same way as before and if $\int_\Omega \mathbf{f} = 0$ then

$$\|\nabla \mathbf{u}_r\|_{\mathbb{L}^2} \leq C\|\mathbf{f}\|_{\mathbb{L}^2}, \ \forall r > 0,$$

where C is a constant independent of r.

To deal with the divergence-free issue, we consider the following divergence problem for $1 < p < \infty$, $g \in L^p(\Omega)$, and $\int_\Omega g = 0$:

$$\begin{cases} \operatorname{div} \mathbf{f} = g & \text{in } \Omega, \\ \mathbf{f} \in \mathbb{W}^{1,p}_0(\Omega). \end{cases} \tag{2.3}$$

When Ω is star-like with respect to every point of $D_R(x_0)$ with $\overline{D}_R(x_0) \subset \Omega$, the existence of a solution $\mathbf{f}$ of this problem is proved in [14, Lem. III.3.1] together with the inequality

$$\|\mathbf{f}\|_{\mathbb{W}^{1,p}_0(\Omega)} \leq C\|g\|_{L^p(\Omega)},$$

where the constant C depends on p, R and the diameter of Ω. Note that the divergence problem does not have a unique solution, since by adding any divergence-free function that vanishes on the boundary to the function $\mathbf{f}$ one would get another solution. Nevertheless, for more general bounded domains, for instance, those satisfying the cone condition, the following result is true (cf. [14, Thm III.3.1, Rmk. III.3.1]).

Theorem 2.6. *Let Ω be a bounded domain in $\mathbb{R}^2$ such that $\Omega = \cup_{j=1}^n U_j$, where each U_j is star-shaped with respect to some open ball B_j with $\overline{B_j} \subset U_j$. Then, given $g \in L^q(\Omega) \cap L^k(\Omega)$, $1 < p, k < \infty$, with $\int_\Omega g = 0$, there exists at least one solution $\mathbf{f} \in \mathbb{W}^{1,q}_0(\Omega) \cap \mathbb{W}^{1,k}_0(\Omega)$ to the divergence problem (2.3) satisfying*

$$\|\mathbf{f}\|_{\mathbb{W}^{1,q}_0(\Omega)} \leq C_q^* C_q \|g\|_{L^q(\Omega)} \qquad \text{and} \qquad \|\mathbf{f}\|_{\mathbb{W}^{1,k}_0(\Omega)} \leq C_k^* C_k \|g\|_{L^k(\Omega)},$$

where C_q and C_k depend on q,k respectively and the diameter of Ω and the smallest radius of the balls B_j. The constant C_q^* is the maximum of

$$C_1 = \left(1 + \frac{|U_1|^{1-1/q}}{|F_1|^{1-1/q}}\right)$$

and

$$C_k = \left(1 + \frac{|U_k|^{1-1/q}}{|F_k|^{1-1/q}}\right) \prod_{i=1}^{k-1} (1 + |F_i|^{1/q-1} |D_i \setminus U_i|^{1-1/q}), \qquad k \geq 2,$$

where $D_i = \cup_{s=i+1}^{n} U_s$ *and* $F_i = U_i \cap D_i$.

We are going to apply this theorem to the domain Ω_ε, see Figure 3.2. In this case, it is not difficult to see that the constant in the inequalities can be bounded independently of ε, as follows. For some $\varepsilon > 0$ consider the domain Ω_ε. U_0 denotes the part enclosed by the dashed lines in the picture, which is a part of the covering. When we perform rotations of $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ of U_0 we obtain a covering of Ω_ε by U_0, U_1, U_2, U_3. As ε decreases the triangle S_0 increases and we can put a fixed ball in S_0 for all smaller ε, such that U_0 is star-like with respect to this ball (we can do the same in each U_i). Moreover, we can easily see that $|F_{01}| = |U_0 \cap U_1|$ can be bounded from below. Therefore, we see that the constants in Theorem 2.6 can be bounded independently of ε, as claimed.

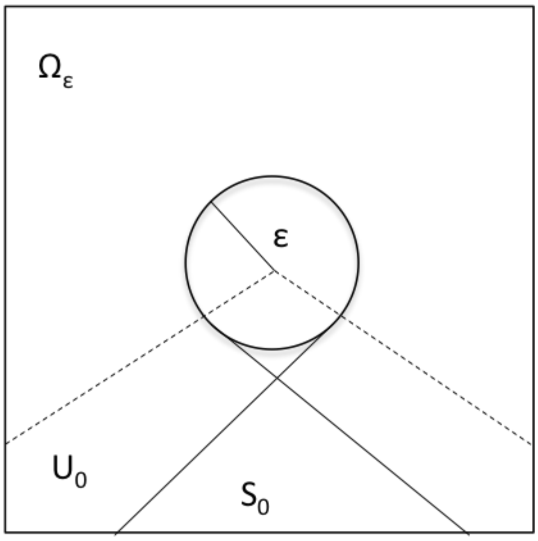


Figure 3.2: The constant in Theorem 2.4 can be taken to be bounded independently of ε.

We now prove the required lemma on the approximation of functions in $\mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$ by functions in $\mathbb{V}_{0,\varepsilon,\sigma}$.

Lemma 2.1. *If* $\mathbf{v} \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$ *then there exists a "sequence"* $\mathbf{v}_\varepsilon \in \mathbb{V}_{0,\varepsilon,\sigma}$ *such that*

$$\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v} \quad \text{in} \quad \mathbb{H}^1(\Omega) \qquad \text{as} \qquad \varepsilon \to 0.$$

Proof. Let $\psi(x) \in C^\infty(\mathbb{R})$ with

$$\psi(x) = \begin{cases} 0 & x \in (-\infty, 1], \\ 1 & x \in [2, \infty). \end{cases}$$

Let $\rho_\varepsilon(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, D_\varepsilon)$, $\mathbf{x} \in \Omega_\varepsilon$ and define $\phi_\varepsilon(\mathbf{x}) \equiv \psi(\frac{1}{\varepsilon}\rho_\varepsilon(\mathbf{x}))$. Then $\phi_\varepsilon = 0$ for $\mathbf{x} \in D_\varepsilon$, $\phi_\varepsilon = 1$ when $\operatorname{dist}(\mathbf{x}, D_\epsilon) \geq 2\varepsilon$, and $\|\nabla\phi_\varepsilon\|_{\mathbb{L}^\infty(\Omega)} \leq \frac{C}{\varepsilon}$ for $0 < \rho_\varepsilon(\mathbf{x}) < 2\varepsilon$.

We first assume that $\mathbf{v} \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega) \cap \mathbb{L}^\infty(\Omega)$. Then for ε small $\phi_\varepsilon \mathbf{v} \in \mathbb{V}_{0,\varepsilon}$. Since $\operatorname{div}(\mathbf{v}) = 0$ it follows

$$\operatorname{div}(\mathbf{v}\phi_\varepsilon) = \nabla\phi_\varepsilon \cdot \mathbf{v}.$$

Moreover,

$$\int\limits_{\Omega_\varepsilon} \nabla\phi_\varepsilon \cdot \mathbf{v} = \int\limits_{\Omega_\varepsilon} \operatorname{div}(\mathbf{v}\phi_\varepsilon) = 0.$$

Noting that also that $\nabla\phi_\varepsilon \cdot \mathbf{v}$ belongs to $\mathbb{L}^\infty(\Omega)$, it follows that it satisfies the conditions required by Theorem 2.6, and so the divergence problem

$$\begin{cases} \operatorname{div} \mathbf{f}_\varepsilon = -\nabla\phi_\varepsilon \cdot \mathbf{v} & \text{in } \Omega_\varepsilon, \\ \mathbf{f}_\varepsilon \in \mathbb{W}^{1,p}_0(\Omega_\varepsilon), \end{cases}$$

has a solution $\mathbf{f}_\varepsilon \in \mathbb{W}^{1,p}_0(\Omega)$, for any $1 < p < \infty$, satisfying

$$\|\mathbf{f}_\varepsilon\|_{\mathbb{W}^{1,p}_0(\Omega_\varepsilon)} \leq C\|\nabla\phi_\varepsilon \cdot \mathbf{v}\|_{L^p(\Omega_\varepsilon)},$$

where C depends only on p and Ω.

Define $\mathbf{v}_\varepsilon := \mathbf{f}_\varepsilon + \phi_\varepsilon \mathbf{v}$, so that $\mathbf{v}_\varepsilon \in \mathbb{V}_{0,\varepsilon,\sigma}$. We will show that $\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v}$ in $\mathbb{H}^1(\Omega)$ as $\varepsilon \to 0$. To this end, observe that

$$\begin{aligned} \|\nabla\phi_\varepsilon \cdot \mathbf{v}\|^p_{L^p(\Omega)} = \int\limits_{\Omega} |\nabla\phi_\varepsilon \cdot \mathbf{v}|^p &= \int\limits_{D_{2\varepsilon}\setminus D_\varepsilon} |\nabla\phi_\varepsilon \cdot \mathbf{v}|^p \\ &\leq \|\nabla\phi_\varepsilon \cdot \mathbf{v}\|^p_{L^\infty(\Omega)} 3\pi\varepsilon^2 \\ &\leq \frac{C^p}{\varepsilon^p}\|\mathbf{v}\|^p_{\mathbb{L}^\infty(\Omega)} 3\pi\varepsilon^2 \end{aligned} \tag{2.4}$$

where C is a constant independent of ε.

Therefore, for $1 < p < 2$, it follows that for some constant C

$$\|\nabla\phi_\varepsilon \cdot \mathbf{v}\|^p_{L^p(\Omega_\varepsilon)} \leq C\varepsilon^{2-p} \to 0$$

as $\varepsilon \to 0$. Hence, we deduce that $\mathbf{f}_\varepsilon$ converges to 0 in $\mathbb{W}^{1,p}_0(\Omega)$ for $1 < p < 2$.

Now, thanks to (2.4) $\mathbf{f}_\varepsilon$ is bounded in $\mathbb{H}^1_0(\Omega)$ and we can extract a "subsequence" that converges weakly to a limit function $\mathbf{f}$ in $\mathbb{H}^1_0(\Omega)$. Since the limit is unique in the distribution sense and $\mathbf{f}_\varepsilon \to 0$ in $\mathbb{W}^{1,p}_0(\Omega)$ we have $\mathbf{f} = 0$ and $\mathbf{f}_\varepsilon \rightharpoonup 0$ in $\mathbb{H}^1_0(\Omega)$.

For every i and j

$$\partial_{x_i}(\phi_\varepsilon v_j) = \phi_\varepsilon \partial_{x_i} v_j + \partial_{x_i}\phi_\varepsilon v_j$$

and arguing as in (2.4) one sees that $\phi_\varepsilon \mathbf{v}$ is bounded in $\mathbb{H}^1(\Omega)$. Moreover, since $\phi_\varepsilon \mathbf{v} \to \mathbf{v}$ in $\mathbb{L}^2(\Omega)$ it follows that $\phi_\varepsilon \mathbf{v} \rightharpoonup \mathbf{v}$ in $\mathbb{H}^1(\Omega)$. Hence, we have found a divergence-free sequence $\mathbf{v}_\varepsilon \in \mathbb{V}_{0,\varepsilon,\sigma}$ that converges weakly to $\mathbf{v}$ in $\mathbb{H}^1(\Omega)$.

It remains only to prove that a function in $\mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$ can be approximated by functions in $\mathbb{H}^1_{\mathrm{per},\sigma}(\Omega) \cap \mathbb{L}^\infty(\Omega)$ which will allow us to conclude via a diagonal argument.

Let $\mathbf{v} \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$. Then there exists a sequence in $\mathbf{w}_n \in \mathbb{C}^1_{\mathrm{per}}(\overline{\Omega})$ such that

$$\mathbf{w}_n \to \mathbf{v} \text{ in } \mathbb{H}^1(\Omega). \tag{2.5}$$

Since $\mathbf{w}_n$ is periodic one has

$$\int\limits_\Omega \operatorname{div}(\mathbf{w}_n - \mathbf{v}) = \int\limits_\Omega \operatorname{div} \mathbf{w}_n = 0.$$

Moreover,

$$\operatorname{div}(\mathbf{w}_n - \mathbf{v}) = \operatorname{div} \mathbf{w}_n \in L^p(\Omega) \quad \forall p \geq 2.$$

Thus, by Theorem 2.6 there exists an $\mathbf{f}_n \in \mathbb{H}^1_0(\Omega) \cap \mathbb{W}^{1,p}_0(\Omega)$ that solves

$$\operatorname{div}(\mathbf{f}_n) = \operatorname{div}(\mathbf{w}_n - \mathbf{v}) \text{ in } \Omega$$

and satisfies the estimates

$$\begin{aligned} &\|\mathbf{f}_n\|_{\mathbb{H}^1_0(\Omega)} \leq C\|\operatorname{div}(\mathbf{w}_n - \mathbf{v})\|_{L^2(\Omega)}, \\ &\|\mathbf{f}_n\|_{\mathbb{W}^{1,p}_0(\Omega)} \leq C\|\operatorname{div}(\mathbf{w}_n)\|_{L^p(\Omega)} \text{ for some } p > 2. \end{aligned}$$

From (2.5) and the first estimate above it follows that $\mathbf{f}_n \to 0$ in $\mathbb{H}^1_0(\Omega)$. The second estimate implies that $\mathbf{f}_n \in \mathbb{L}^\infty(\Omega)$, since $\mathbb{W}^{1,p}_0(\Omega)$ is embedded into $\mathbb{L}^\infty(\Omega)$ for $p > 2$. Now if we set

$$\mathbf{v}_n = \mathbf{w}_n - \mathbf{f}_n,$$

then clearly $\mathbf{v}_n \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega) \cap \mathbb{L}^\infty(\Omega)$ and $\mathbf{v}_n \to \mathbf{v}$ in $\mathbb{H}^1(\Omega)$. □

To prove Theorem 2.5 we essentially recapitulate the proof of Theorem 1.4 in this new setting.

Proof. (Theorem 2.5) Define

$$\tilde{\mathbf{u}}_r = \mathbf{u}_r - ⨍_\Omega \mathbf{u}_r.$$

Then from the Poincaré inequality, $\|\tilde{\mathbf{u}}_r\|_{\mathbb{H}^1(\Omega_r)}$ is uniformly bounded. Therefore for a subsequence $\nabla \mathbf{u}_r = \nabla \tilde{\mathbf{u}}_r \rightharpoonup \nabla \mathbf{u}_0$ in $\mathbb{H}^1(\Omega)$ and $\tilde{\mathbf{u}}_r \to \mathbf{u}_0$ in $\mathbb{L}^2(\Omega)$, where $\mathbf{u}_0$ satisfies $\int_\Omega \mathbf{u}_0 = 0$.

For a fixed r_0, $\forall r < r_0$ one has $\mathbb{V}_{0,\sigma,r_0} \subset \mathbb{V}_{0,\sigma,r}$. Thus

$$\int_\Omega \nabla \mathbf{u}_r : \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \text{for all } \mathbf{v} \in \mathbb{V}_{0,\sigma,r_0}.$$

Passing to the limit in r we obtain

$$\int_\Omega \nabla \mathbf{u}_0 : \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \text{for all } \mathbf{v} \in \mathbb{V}_{0,\sigma,r_0}. \tag{2.6}$$

Let $\mathbf{v} \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$ and let $\mathbf{v}_\varepsilon$ be the approximating sequence from Lemma 2.1. Then for $\varepsilon \leq r_0$ we have

$$\int_\Omega \nabla \mathbf{u}_0 : \nabla \mathbf{v}_\varepsilon = \int_\Omega \mathbf{f} \cdot \mathbf{v}_\varepsilon$$

and passing to the limit in ε we obtain

$$\int_\Omega \nabla \mathbf{u}_0 : \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \text{for all } \mathbf{v} \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$$

as required. (This is (2.2).)

Since the limiting problem has a unique solution when one imposes the zero average condition, it follows that all convergent “subsequences” must have the same limit. As a consequence, the whole original “sequence” converges toward $\mathbf{u}_0$.

To see that $\nabla \mathbf{u}_r \to \nabla \mathbf{u}_0$ in $\mathbb{L}^2(\Omega)$ we show that $\|\nabla \mathbf{u}_r\|^2_{\mathbb{L}^2(\Omega)} \to \|\nabla \mathbf{u}_0\|^2_{\mathbb{L}^2(\Omega)}$. Since $\mathbf{u}_r - \fint \mathbf{u}_r \to \mathbf{u}_0$ in $\mathbb{L}^2(\Omega)$,

$$\int_{\Omega_r} |\nabla \mathbf{u}_r|^2 = \int_{\Omega_r} \mathbf{f} \cdot \mathbf{u}_r = \int_\Omega \mathbf{f} \cdot \mathbf{u}_r = \int_\Omega \mathbf{f} \cdot \left(\mathbf{u}_r - \fint \mathbf{u}_r \right) \to \int_\Omega \mathbf{f} \cdot \mathbf{u}_0.$$

But from (2.2) we have

$$\int_\Omega |\nabla \mathbf{u}_0|^2 = \int_\Omega \mathbf{f} \cdot \mathbf{u}_0,$$

which implies that

$$\int_\Omega |\nabla \mathbf{u}_r|^2 \to \int_\Omega |\nabla \mathbf{u}_0|^2.$$

Coupled with weak convergence this implies strong convergence of $\nabla \mathbf{u}_r$ to $\nabla \mathbf{u}_0$ in $\mathbb{L}^2(\Omega)$. $\square$

3 The stationary Navier–Stokes equations

The next natural problem to consider is the stationary Navier–Stokes system

$$\begin{cases} -\Delta \mathbf{u}_r + (\mathbf{u}_r \cdot \nabla)\mathbf{u}_r + \nabla p_r = \mathbf{f} \ \text{ in } \Omega_r, \\ \operatorname{div} \mathbf{u}_r = 0 \text{ in } \Omega_r, \\ \mathbf{u}_r = 0 \text{ in } \partial D_r, \\ \mathbf{u}_r \text{ periodic.} \end{cases} \tag{3.1}$$

However, there are some differences between this model and the others that we considered in this thesis, which arise essentially because when $\mathbf{f}$ is large (in some appropriate sense) we cannot guarantee the existence of a unique solution of (3.1), nor of the limiting problem

$$\begin{cases} -\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \ \text{ in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \\ \mathbf{u} \text{ periodic.} \end{cases} \tag{3.2}$$

We note immediately that as with the Laplace/Stokes problems, this equation can only have a solution if $\int_\Omega \mathbf{f} = 0$. This is easily seen, formally, if one writes the nonlinear term as $\nabla\cdot(\mathbf{u}\otimes\mathbf{u})$ and then simply integrates over Ω using the periodic boundary conditions.

To see that there is non-uniqueness, and that there is no guarantee in general that $\int_\Omega \mathbf{u} = 0$ (when $\mathbf{u}$ solves (3.2)), take $\varphi \in C^\infty([0,4]) \to \mathbb{R}$ with the property that $\varphi((2-t)^2) = \varphi(t^2)$. For $\mathbf{x} \in \Omega = (-1,1)^2$, let $v(x_1,x_2) = \varphi((x_1-x_2)^2)$. Note that v is a periodic function of $\mathbf{x}$:

$$v(x_1,1) = \varphi((-1+x_1)^2) = \varphi((1+x_1)^2) = v(x_1,-1)$$

and similarly $v(1,x_2) = v(-1,x_2)$. Now, define a vector field $\mathbf{u}$ by setting

$$\mathbf{u}(x_1,x_2) = (v(x_1,x_2), v(x_1,x_2))$$

This field has first derivatives

$$\partial_1 u_j = \partial_1 v = 2(x_1-x_2)\varphi'((x_1-x_2)^2) \qquad \text{and} \qquad \partial_2 u_j = -\partial_1 u_j.$$

So

$$\nabla\cdot\mathbf{u} = 0, \qquad (\mathbf{u}\cdot\nabla)\mathbf{u} = 0, \qquad (\mathbf{c}\cdot\nabla)\mathbf{u} = 0,$$

where $\mathbf{c} = (1,1)$.

Now let $\mathbf{f} = -\Delta\mathbf{u}$. Since $(\mathbf{u}\cdot\nabla)\mathbf{u} = 0$ and $\mathbf{u}$ is divergence-free, it follows that $\mathbf{u}$ is a solution of (3.2). However, $\mathbf{u}+\alpha\mathbf{c}$ is also a solution for any $\alpha \in \mathbb{R}$. In particular, one can ensure that $\int_\Omega \mathbf{u}$ takes any chosen value in the direction of $\mathbf{c}$.

However, note that in this example there is a unique solution $\mathbf{u}$ determined by the condition that $\int_\Omega \mathbf{u} = 0$.

Let $\mathbf{f} \in \mathbb{L}^2(\Omega)$ and let $\mathbf{u}_r \in \mathbb{V}_{0,r,\sigma}$ be the solution of the problem

$$\int_{\Omega_r} \nabla\mathbf{u}_r : \nabla\mathbf{v} + \int_{\Omega_r} [(\mathbf{u}_r\cdot\nabla)\mathbf{u}_r]\cdot\mathbf{v} = \int_{\Omega_r} \mathbf{f}\cdot\mathbf{v} \qquad \text{for all } \mathbf{v} \in \mathbb{V}_{0,r,\sigma}. \tag{3.3}$$

Since $\mathbf{u}_r \in \mathbb{V}_{0,r,\sigma}$ we can take $\mathbf{v} = \mathbf{u}_r$ in this identity to obtain

$$\int_{\Omega_r} \|\nabla \mathbf{u}_r\|^2 = \int_{\Omega_r} \mathbf{f} \cdot \mathbf{u}_r = \int_{\Omega} \mathbf{f} \cdot (\mathbf{u}_r - \langle \mathbf{u}_r \rangle),$$

where as before we have also used the fact that $\int_\Omega \mathbf{f} = 0$. It follows that

$$\|\nabla \mathbf{u}_r\|^2 \leq \|\mathbf{f}\| \, \|\mathbf{u}_r - \langle \mathbf{u}_r \rangle\| \leq C\|\mathbf{f}\|\|\nabla \mathbf{u}_r\|,$$

and so $\nabla \mathbf{u}_r$ and $\tilde{\mathbf{u}}_r = \mathbf{u}_r - \langle \mathbf{u}_r \rangle$ are uniformly bounded in $L^2(\Omega)$. As before, $\nabla \tilde{\mathbf{u}}_r \rightharpoonup \nabla \mathbf{u}$ weakly in L^2 and $\tilde{\mathbf{u}}_r \to \mathbf{u}$ strongly in L^2.

Now, if we rewrite (3.3) in terms of $\tilde{\mathbf{u}}_r$, rearrange, and integrate by parts then we obtain

$$\int_{\Omega_r} [(\langle \tilde{\mathbf{u}}_r \rangle \cdot \nabla)\tilde{\mathbf{u}}_r] \cdot \mathbf{v} = -\int_{\Omega_r} \nabla \tilde{\mathbf{u}}_r : \nabla \mathbf{v} + \int_{\Omega_r} [(\tilde{\mathbf{u}}_r \cdot \nabla)\mathbf{v}] \cdot \tilde{\mathbf{u}}_r + \int_{\Omega_r} \mathbf{f} \cdot \mathbf{v}$$

for all $\mathbf{v} \in \mathbb{V}_{0,r,\sigma}$. Using the same trick as before (since $\int \mathbf{f} = 0$) we can bound the right hand side uniformly in r,

$$\begin{aligned}\left| \int_{\Omega_r} [(\langle \tilde{\mathbf{u}}_r \rangle \cdot \nabla)\tilde{\mathbf{u}}_r] \cdot \mathbf{v} \right| &\leq \|\nabla \tilde{\mathbf{u}}_r\|\|\nabla \mathbf{v}\| + \|\nabla \mathbf{v}\|\|\tilde{\mathbf{u}}_r\|_{L^4}^2 + \|\mathbf{f}\|\|\nabla \mathbf{v}\| \\ &= \left\{ \|\nabla \tilde{\mathbf{u}}_r\| + \|\tilde{\mathbf{u}}_r\|_{L^4}^2 + \|\mathbf{f}\| \right\} \|\nabla \mathbf{v}\| \\ &\leq \{ \|\nabla \tilde{\mathbf{u}}_r\| + \|\tilde{\mathbf{u}}_r\|\|\nabla \mathbf{u}_r\| + \|\mathbf{f}\| \} \|\nabla \mathbf{v}\|.\end{aligned}$$

Now, an integration by parts yields

$$\left| \int_{\Omega_r} [(\langle \tilde{\mathbf{u}}_r \rangle \cdot \nabla)\mathbf{v}] \cdot \tilde{\mathbf{u}}_r \right| \leq C\|\nabla \mathbf{v}\|.$$

Taking the inner product of (3.1) with $\mathbf{u}_r$ yields, as with the Laplace/Stokes problem,

$$\|\nabla \mathbf{u}_r\|_{L^2}^2 = \int_\Omega \mathbf{f} \cdot (\mathbf{u}_r - \fint_\Omega \mathbf{u}_r) \leq \|\mathbf{f}\|_{L^2}\|\nabla \mathbf{u}_r\|_{L^2},$$

and hence a bound on $\|\nabla \mathbf{u}_r\|_{L^2}$ that is uniform in r. However, once again we do not obtain a uniform bound on $\mathbf{u}_r$ in L^2, but only on $\tilde{\mathbf{u}}_r = \mathbf{u}_r - \fint_\Omega \mathbf{u}_r$.

However, if we rewrite (3.1) as an equation for $\tilde{\mathbf{u}}_r$ then we obtain

$$-\Delta \tilde{\mathbf{u}}_r + (\tilde{\mathbf{u}}_r \cdot \nabla)\tilde{\mathbf{u}}_r + (\langle \mathbf{u}_r \rangle \cdot \nabla)\tilde{\mathbf{u}}_r + \nabla p_r = \mathbf{f},$$

where $\langle \mathbf{u}_r \rangle = \fint_\Omega \mathbf{u}_r$. Thus, because of the non-linear term we cannot deduce the uniform bound in H^1 as for the Stokes case.

3.1 Existence

Nevertheless, we could use the same technique as in Chapter 2 to establish the existence result in each punctured domain Ω_r. Let

$$\dot{\mathbb{V}}_{0,r,\sigma} = \{\mathbf{u}_r \in \mathbb{V}_{0,r,\sigma} : \int\limits_{\Omega} \mathbf{u}_r = 0\}.$$

Find $\mathbf{u}_r \in \dot{\mathbb{V}}_{0,r,\sigma}$ such that

$$-((\mathbf{u}_r \cdot \nabla)\mathbf{v}, \mathbf{u}_r) + \nu(\nabla \mathbf{u}_r, \nabla \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad \mathbf{v} \in \mathbb{V}_{0,r,\sigma} \tag{3.4}$$

holds in the weak formulation. We assume that $\int_{\Omega_r} \mathbf{f} \cdot \mathbf{v}\, dx$ defines a linear functional of $\mathbf{v} \in \dot{\mathbb{V}}_{0,r,\sigma}$. We want to apply the Leray–Schauder fixed point theorem to establish the existence of solutions. First, note that

$$((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}) = -((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{u}), \quad \forall \mathbf{u}, \mathbf{v} \in \dot{\mathbb{V}}_{0,r,\sigma} \tag{3.5}$$

holds. To see that, notice

$$((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}) = \int\limits_{\partial\Omega_r} (\mathbf{u} \cdot \mathbf{n})(\mathbf{u} \cdot \mathbf{v}) d\sigma - \int\limits_{\Omega} (\mathbf{u} \cdot \mathbf{v}) \operatorname{div} \mathbf{u}\, dx - ((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{u}).$$

Because of the periodic condition of $\mathbf{u}$ the first term of the RHS is equal to zero. The second term equals to zero because $\mathbf{u}$ is solenoidal.
Since $\int_{\Omega_r} \mathbf{f} \cdot \mathbf{v}\, dx$ defines a linear functional of $\mathbf{v} \in \dot{\mathbb{V}}_{0,r,\sigma}$, there exists a unique $\tilde{\mathbf{f}} \in \dot{\mathbb{V}}_{0,r,\sigma}$ such that

$$\int\limits_{\Omega_r} \mathbf{f} \cdot \mathbf{v}\, dx = \langle \tilde{\mathbf{f}}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \dot{\mathbb{V}}_{0,r,\sigma}.$$

$((\mathbf{u}_r \cdot \nabla)\mathbf{v}, \mathbf{u}_r)$ defines also a linear functional of $\mathbf{v} \in \dot{\mathbb{V}}_{0,r,\sigma}$. The boundedness of the functional can be showed as follows,

$$\begin{aligned}
|((\mathbf{u}_r \cdot \nabla)\mathbf{v}, \mathbf{u}_r)| &\le c\|\mathbf{u}_r\|^2_{L^4(\Omega_r)} \|\nabla \mathbf{v}\| \\
&\le C\|\mathbf{u}_r\|^2_{L^4(\Omega)} \|\nabla \mathbf{v}\| \\
&\le C\|\nabla \mathbf{u}_r\|^2 \|\nabla \mathbf{v}\|.
\end{aligned}$$

Thus, for every fixed $\mathbf{u}_r \in \dot{\mathbb{V}}_{0,r,\sigma}$ the map $\mathbf{v} \mapsto ((\mathbf{u}_r \cdot \nabla)\mathbf{v}, \mathbf{u}_r)$ is a bounded linear map on $\mathbb{V}_{0,r,\sigma}$. It follows from the Riesz Representation Theorem that there exists an element $\mathbf{B}(\mathbf{u}_r) \in \dot{\mathbb{V}}_{0,r,\sigma}$ such that

$$\langle \mathbf{B}(\mathbf{u}_r), \mathbf{v} \rangle = \langle (\mathbf{u}_r \cdot \nabla)\mathbf{v}, \mathbf{u}_r \rangle \qquad \text{for all} \quad \mathbf{v} \in \mathbb{V}_{0,r,\sigma}.$$

Thus $\mathbf{u}_r$ satisfies (3.4) if and only if

$$-\mathbf{B}(\mathbf{u}_r) + \nu \mathbf{u}_r - \tilde{\mathbf{f}} = 0.$$

We show that $\mathbf{B}$ is completely continuous in $\dot{\mathbb{V}}_{0,r,\sigma}$.

Let $(\mathbf{u})_n$ be a weakly convergent sequence in $\dot{\mathbb{V}}_{0,r,\sigma}$. Since $\dot{\mathbb{V}}_{0,r,\sigma}$ can be compactly embedded in $\mathbb{L}^4(\Omega_r)$, $(\mathbf{u})_n$ converges strongly in $\mathbb{L}^4(\Omega_r)$ to the limit $\mathbf{u}_r$. Therefore

$$\begin{aligned}
&\langle \mathbf{B}(\mathbf{u}_m) - \mathbf{B}(\mathbf{u}_n), \mathbf{v}\rangle = \textstyle\int_{\Omega_r}(\mathbf{u}_m \cdot \nabla)\mathbf{v} \cdot \mathbf{u}_m) - \int_{\Omega_r}(\mathbf{u}_n \cdot \nabla)\mathbf{v} \cdot \mathbf{u}_n \\
&= \textstyle\int_{\Omega_r}((\mathbf{u}_m - \mathbf{u}_n) \cdot \nabla)\mathbf{v} \cdot \mathbf{u}_m) + \int_{\Omega_r}(\mathbf{u}_n \cdot \nabla)\mathbf{v} \cdot (\mathbf{u}_m - \mathbf{u}_n) \qquad (3.6) \\
&\leq C_1\|\mathbf{u}_m - \mathbf{u}_n\|_{L^4}(\|\nabla \mathbf{u}_m\| + \|\nabla \mathbf{u}_n\|)\|\nabla \mathbf{v}\|.
\end{aligned}$$

Since the weakly convergent sequence is bounded and when we set

$$\mathbf{v} = \mathbf{B}(\mathbf{u}_m) - \mathbf{B}(\mathbf{u}_n)$$

into the above equation it follows that

$$\|\nabla(\mathbf{B}(\mathbf{u}_m) - \mathbf{B}(\mathbf{u}_n))\| \leq C_2\|\mathbf{u}_m - \mathbf{u}_n\|_{L^4} \to 0 \text{ as } m, n \to \infty.$$

This operator equation has at least one solution as showed in Chapter 2 Section **The Boundary Value Problem in Bounded Domain** in this thesis.

4 Conclusions

We have analysed three models in a simple but unusual geometry, the 'punctured periodic domain', showing that the influence of the obstacle, a disc of radius r, evaporates in the limit as $r \to 0$.

Some interesting open problems remain. While the lack of a bound on the average of the solution u_r over Ω (in both the Poisson and Stokes problems) that is uniform in r appears initially to be only a mathematical curiosity, such a bound is central to tackling the stationary Navier–Stokes problem in this geometry.

The fact that there is no 'uniform elliptic regularity' for the Laplacian or Stokes operator in this geometry means that the important 'vanishing tracer' problem (cf. [8, 55]) also remains open.

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